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SPINS AND BARYONS IN THE TOPOLOGICAL EXPANSION

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April 1980
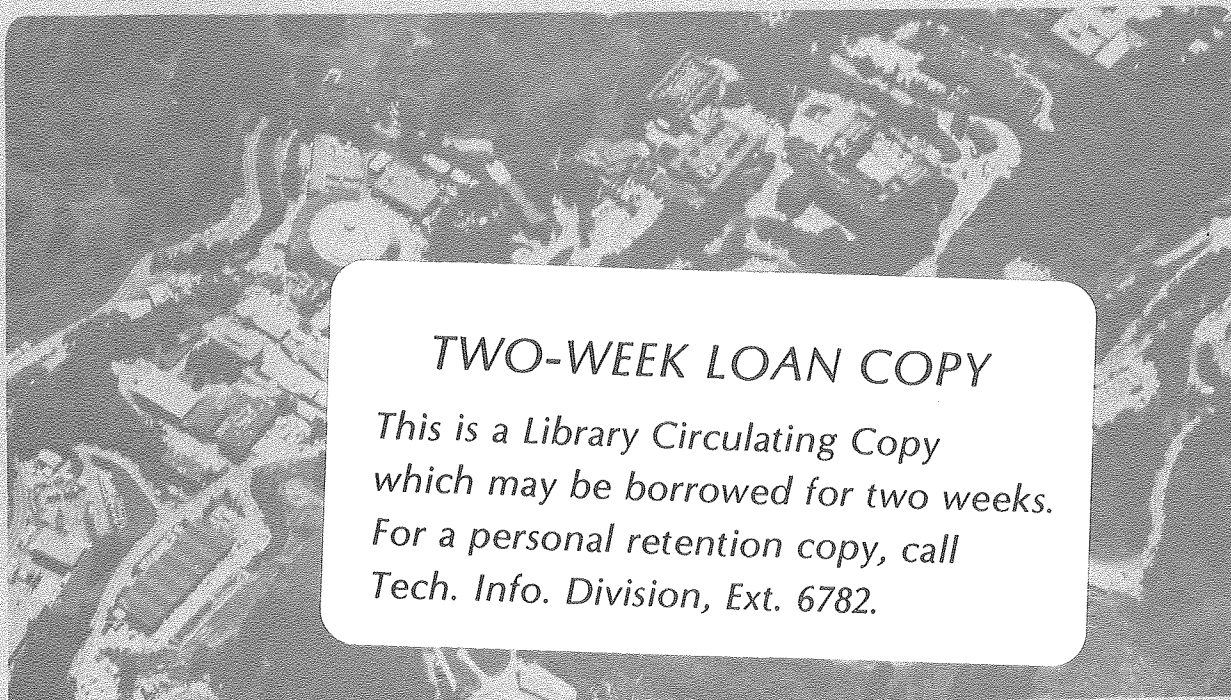
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SPINS AND BARYONS IN THE

TOPOLOGICAL EXPANSION

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April 5, 1980

ABSTRACT

Spins, baryons, and the group-theoretic structure
of the constituent quark model are incorporated into
the topological expansion scheme of hadron physics.

## 1. INTRODUCTION

A scattering amplitude can be represented as a sum of contributions from all ways in which the process can occur. Each contribution has a phase factor, and the scattering amplitude between randomly chosen states tends to be small due to an averaging-out of these phase factors. The dominant transitions are between states in which the elements of order characterizing the initial state are carried into the final state in some "direct" way.

This tendency of the the dominant transitions to preserve order is particularly important in hadron physics, due to the inherent complexity of the hadrons and their interactions. Indeed, this order-preserving tendency has been made the basis of a successful approximation procedure for meson physics. This procedure is based not on the smallness of any coupling constant but rather on the smallness of contributions that do not preserve order. Order is defined so that it is preserved by contributions to the scattering amplitude that correspond to sequences of scattering events represented by graphs that can be drawn in a plane with no lines crossing. Contributions from non planar graphs generally have phase factors that tend to average to zero in high-energy regimes.

This topological approach to hadron dynamics, which originated in some works by Veneziano[1], and has been pursued by many workers, has been recently reviewed by Chew and Rosenzweig[2]. They show how the topological expansion procedure, combined with the requirements of unitarity, analyticity, duality, and Lorentz invariance, organizes and predicts many of the dominant features of meson physics.

The two major deficiencies in the theory described by Chew and Rosenzweig are the omission of spins and baryons. The aim of the present work is to complete the theory by incorporating these two elements. The group-theoretic properties of the constituent-quark model of hadrons are also incorporated. Thus the leading baryons constructed from three kinds of flavors fall into the familar $(56, 0^+)$ and $(70, 1^-)$ multiplets.

The theory is formulated completely within the S-matrix framework, and involves no microscopic description of the hadrons in terms of quark wave functions. Thus it provides a covariant approach to hadron physics that incorporates the group-theoretic properties of the constitutent-quark model and has no confinement problem.

The theory is the product of a long intermittent collaboration with Geoffrey Chew, and his ideas are woven into it in many ways. The technical formulations are of my own making, but the general strategy incorporates key suggestions by Chew.

The present paper is associated with a recent series of papers by Chew and Poénaru.[3] It describes technical results that have been used in the development of their ideas. However, the aims of Chew and Poénaru are broader than those of the present work, which simply accepts the group-theoretic structure of the constituent-quark model on the basis of its empirical success. Chew and Poénaru seek to derive the group-theoretic structures from topological considerations and consequently need a richer topological structure than the one used here. Their topological structure contains, in addition to the quark-particle graphs of the present theory, and

surface upon which these graphs are imbedded, also a second surface, called the quantum surface, in which the group-theoretic relations associated with flavor and other symmetries reside.

In the present work flavor is an unconstrained variable. The flavor structure may in fact be determined by the nonlinear dynamical equations, but it is not determined within the present framework by topological considerations alone.

The theory is based on the covariant treatment of spin provided by the M function formalism. Since the earlier description of this formalism[4] was very brief the key points are described here in §2, with particular emphasis on those results that are important in the context of the present work.

The incorporation of spin into the meson sector is described in §3. It has two principal innovations, compared to earlier efforts in this direction[5]. The first is a novel way of incorporating parity conservation. This procedure allows for effective parity doubling in the intermediate states at the lowest level of the topological expansion scheme, although the unphysical-parity particles are present neither as external particles at any level, nor as internal particles in the full theory.

These effective parity doublets are artifacts of an averaging over phase factors $(-1)^\ell$ that suppresses certain averaged contributions, and justifies the classification of these contributions as higher-order in the topological expansion. However, when individual $\ell$ values are considered this averaging is not possible, and the lowest-order approximation is inadequate.

The second novel feature in the meson sector is the absence
of $\rho\pi\pi$ coupling in lowest order of the topological expansion. This
resonance is quite narrow, and $\rho\pi\pi$ coupling is considered a higher-
order effect.

Baryons are treated in §4. The topological structure is
essentially the same as in the meson sector. This is achieved
by treating the baryon at the lowest-order (zero-entropy) level of
the topological expansion as a quark-diquark combination. The novel
feature in the baryon sector is the introduction of a two-dimensional
representation of the permutation group $S_3$ in association with
each of the vector indices $\mu_i$ that arise in connection with the
Regge recurrences of the baryons. The physical amplitudes are
required to be invariant under all permutations of the group $S_3$,
applied separately to each baryon. This imposes a full permutation
symmetry analogous to that of the constituent-quark model, and leads
to the familiar $\ell = 0$ and $\ell = 1$ multiplets.

An appendix contains a brief description of the reasons for the
failure of earlier attempts[6-10] to incorporate baryons into the topo-
logical expansion.

## 2.  SPIN

### 2.1.  Lorentz Transformations in Spin Space

Let $\sigma_\mu$ represent the Pauli spin-matrix four-vector

$$\sigma_\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (1, \vec{\sigma}), \qquad (2.1)$$

where $\sigma_0$ is the two-by-two unit matrix and $\sigma_1$, $\sigma_2$, and $\sigma_3$ are

the three Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.2)$$

Let $A$ and $B$ be any two-by-two matrices with determinant one.
Then the Lorentz transformation matrix $L^\mu{}_\nu(A,B)$ is defined by

$$A\sigma_\mu B = \sigma_\nu L^\nu{}_\mu(A,B) \equiv (\sigma \cdot L)_\mu \qquad (2.3)$$

(Repeated vector and spinor indices are always to be summed.)

Let $\tilde{\sigma}_\mu$ represent the Pauli spin-matrix four vector

$$\tilde{\sigma}_\mu = (1, -\vec{\sigma}), \qquad (2.4)$$

Then

$$\frac{1}{2} \text{Tr } \sigma_\mu \tilde{\sigma}_\nu = g_{\mu\nu}, \qquad (2.5)$$

where $g_{\mu\nu}$ is the Lorentz metric tensor with diagonal elements
$(1, -1, -1, -1)$.

Let $C = -i\sigma_2 = -C^{Tr}$ be the (charge) conjugation matrix,
and let $M$ be any two-by-two matrix. Then the Pauli identity

$$C^{-1} M^{Tr} C M = \det M, \qquad (2.6)$$

entails that

$$C^{-1}\sigma_\mu^{Tr} C = \tilde{\sigma}_\mu \qquad (2.7)$$

and that

$$B^{-1} \tilde{\sigma}_\mu A^{-1} = \tilde{\sigma}_\nu \ L^\nu_{\ \mu}(A,B) \tag{2.8}$$

$$\equiv (\tilde{\sigma} \cdot L)_\mu \ .$$

To specify four different ways of applying transforms to spin indices four different types of spinor indices are introduced. The spin transformation $\Lambda \equiv \Lambda(A,B)$ acts on the different types of spinor indices according to the rules:

$$\Lambda \ (\phi_\alpha) = A_\alpha^{\ \alpha'} \ \phi_{\alpha'} \equiv (A\phi)_\alpha \tag{2.9}$$

$$\Lambda \ (\phi_{\dot\beta}) = \phi_{\dot\beta'} \ B^{\dot\beta'}_{\ \dot\beta} = (\phi B)_{\dot\beta}$$

$$\Lambda \ (\phi^{\dot\beta}) = (B^{-1})^{\dot\beta}_{\ \dot\beta'} \phi^{\dot\beta'} = (B^{-1}\phi)^{\dot\beta}$$

$$\Lambda \ (\phi^\alpha) = \phi^{\alpha'} (A^{-1})_{\alpha'}^{\ \alpha} = (\phi A^{-1})^\alpha \ .$$

Thus the transformation to be applied is determined by the location of the index (upper or lower) and whether it is dotted or undotted.

The operator $\Lambda$ acts like the identity on any sum of the form $\phi^\alpha \psi_\alpha$ or $\phi_{\dot\beta} \psi^{\dot\beta}$ . For example,

$$\Lambda \ (\phi^\alpha \psi_\alpha) = (\Lambda\phi^\alpha)(\Lambda\psi_\alpha)$$

$$= (\phi^{\alpha'} A_{\alpha'}^{-1\alpha})(A_\alpha^{\ \alpha''} \psi_{\alpha''})$$

$$= \phi^\alpha \psi_\alpha \ . \tag{2.10}$$

Let $a_1, a_2, \ldots, a_{2n}$ be any set of $2n$ four-vectors. Then

$$\frac{1}{2} \operatorname{Tr} a_1 \cdot \sigma \ a_2 \cdot \tilde{\sigma} \ a_3 \cdot \sigma \ \ldots \ a_{2n} \cdot \tilde{\sigma} \tag{2.11}$$

is a Lorentz-invariant function of the four-vectors $a_1, \ldots a_{2n}$. To see this let the indices on $\sigma_\mu$ and $\tilde{\sigma}_\mu$ be specified always in the following way:

$$\sigma_\mu \to \sigma_{\mu\alpha\dot\beta} \qquad \tilde{\sigma}_\mu \to \sigma_\mu^{\dot\beta\alpha} \ . \tag{2.12a}$$

Then (2.3) and (2.8) become

$$\Lambda\sigma_\mu = (\sigma \cdot L)_\mu \qquad \Lambda\tilde{\sigma}_\mu = (\tilde{\sigma} \cdot L)_\mu . \tag{2.12b}$$

Application of the operator $\Lambda$ leaves invariant the trace (2.11), due to (2.10). It gives, alternatively, by virtue of (2.12),

$$\frac{1}{2} \operatorname{Tr} (\sigma \cdot La_1) (\tilde{\sigma}_\nu \cdot La_2) \ldots (\tilde{\sigma}_{2n} \cdot La_{2n}). \tag{2.13}$$

Thus the trace is invariant under any Lorentz transformation of all the vectors $a_i$.

Two important special cases are

$$\frac{1}{2} \operatorname{Tr} a_1 \cdot \sigma \ a_2 \cdot \tilde{\sigma} = a_1 \cdot a_2 \ , \tag{2.14a}$$

which follows from (2.5), and

$$\frac{1}{2} \text{Tr } a_1 \cdot \sigma \;\; a_2 \cdot \tilde{\sigma} \;\; a_3 \cdot \sigma \;\; a_4 \cdot \tilde{\sigma}$$

$$= (a_1 \cdot a_2)(a_3 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3) - (a_1 \cdot a_3)(a_2 \cdot a_4)$$

$$+ i [a_1, a_2, a_3, a_4] , \qquad (2.14b)$$

where

$$[a_1, a_2, a_3, a_4] = a_1^{\mu} a_2^{\nu} a_3^{\sigma} a_4^{\delta} \; \varepsilon_{\mu\nu\sigma\delta} \qquad . \qquad (2.15)$$

Here $\varepsilon$ is the fully antisymmetric matrix with $\varepsilon_{0123} = 1$.

## 2.2 Covariant Spin-Projection Operators

Let $p = mv$ be the momentum-energy of a freely moving particle, as measured in some general Lorentz frame $\Sigma$. Let $s$ be a spin vector that satisfies $s \cdot p = 0$. Let $\Sigma^r(v)$ be the particle-rest-frame obtained by applying a "boost" to $\Sigma$. This boost is a Lorentz transformation that leaves unchanged any space component that is perpendicular to $\vec{v}$. The vectors $v$ and $s$ as measured in $\Sigma^r(v)$ are

$$v^r = (v^{r\mu}) \equiv (v^{ro}, \vec{v}^r) = (1, 0, 0, 0) \qquad (2.16a)$$

and

$$s^r = (s^r_{\mu}) \equiv (s^{ro}, \vec{s}^r) = (0, s^r_1, s^r_2, s^r_3) \qquad (2.16b)$$

The rest-frame projection operator is

$$\tilde{P}^r(s) = \frac{1}{2} (1 + \vec{s}^r \cdot \vec{\sigma})$$

$$= \frac{1}{2} (v^r + s^r)^{\mu} \tilde{\sigma}_{\mu} = \frac{1}{2} (v^r + s^r) \cdot \tilde{\sigma}. \qquad (2.17)$$

This operator projects onto the spin state in which the spin is directed along $s^r = (0, \vec{s}^r)$ as measured in $\Sigma^r(v)$, and hence along $s$ as measured in $\Sigma$.

The operator $\tilde{P}^r(s)$ refers to the rest frame $\Sigma^r(v)$. To eliminate this frame dependence one may apply the boost $\Lambda(A,B)$ that converts $\tilde{P}^r$ from its form in $\Sigma^r(v)$ to its form in the general coordinate frame $\Sigma$:

$$\tilde{P}^r(s) \rightarrow \Lambda(v) \; \tilde{P}^{(r)}(s)$$

$$= B^{-1} \tilde{P}^{(r)}(s) \; A^{-1}$$

$$= \frac{1}{2} \tilde{\sigma} \cdot (Lv^r + Ls^r)$$

$$= \frac{1}{2} \tilde{\sigma} \cdot (v + s)$$

$$= \frac{1}{2} (v + s) \cdot \tilde{\sigma}$$

$$\equiv \tilde{P}(s, v) \quad . \qquad (2.18)$$

Real Lorentz transformations are generated by matrices $A$ and $B$ that satisfy $A = B^{\dagger}$, where dagger denotes hermitian conjugation.

For rotations $A$ is unitary, but for boosts $A$ is hermitian. The boost $\Lambda(A,B)$ that converts the rest frame form $\tilde{P}^r$ into the general coordinate system form $\tilde{P}$ is

$$A(v) = B^\dagger(v) = A^\dagger(v) = \sqrt{v \cdot \sigma} \qquad (2.19a)$$

$$A^{-1}(v) = B^{\dagger -1}(v) = B^{-1}(v) = \sqrt{v \cdot \tilde{\sigma}} \qquad (2.19b)$$

where

$$\sqrt{v \cdot \sigma} = \exp \frac{\theta}{2} (\vec{\sigma} \cdot \vec{n}) = \cosh \frac{\theta}{2} + \vec{n} \cdot \vec{\sigma} \sinh \frac{\theta}{2} \qquad (2.20)$$

and

$$v \cdot \sigma = \exp \theta(\vec{\sigma} \cdot \vec{n}) = \cosh \theta + \vec{n} \cdot \vec{\sigma} \sinh \theta$$

$$= v^\mu \sigma_\mu$$

$$\equiv v^0 + \vec{v} \cdot \vec{\sigma}$$

$$\equiv v^0 + \vec{n} \cdot \vec{\sigma} |\vec{v}| \qquad (2.21)$$

Note that

$$v \cdot \sigma \quad v \cdot \tilde{\sigma} = 1 \qquad (2.22a)$$

and

$$\sqrt{v \cdot \sigma} \quad \sqrt{v \cdot \tilde{\sigma}} = 1 \quad . \qquad (2.22b)$$

Another useful form is

$$\sqrt{v \cdot \sigma} = \frac{v_0 + 1 + \vec{v} \cdot \vec{\sigma}}{(2v_0 + 2)^{\frac{1}{2}}} \quad . \qquad (2.23)$$

The operator

$$\tilde{P}(s,v) = \sqrt{v \cdot \tilde{\sigma}} \; \frac{1}{2} (1 + \vec{s}^r \cdot \vec{\sigma}) \sqrt{v \cdot \tilde{\sigma}} \qquad (2.24)$$

$$= \frac{1}{2} (v \cdot \tilde{\sigma} + s \cdot \tilde{\sigma})$$

is called a covariant spin operator. The vectors $v$ and $s$ occurring in $\tilde{P}(s \cdot v)$ have components $v^\mu$ and $s^\mu$ that refer to the general frame of reference $\Sigma$.

Because the boost operators $A^{-1}$ and $B^{-1}$ are hermitian, rather than unitary, the operator $\tilde{P}(s, v)$ is not a true projection operator: $\tilde{P}(s, v)^2 \neq \tilde{P}(s, v)$ for $\vec{v} \neq 0$.

The covariant spin operators are Lorentz invariant spinor functions in the sense that

$$\Lambda \tilde{P}(L^{-1}s, L^{-1}v) = \tilde{P}(s, v) . \qquad (2.25)$$

Here $\Lambda = \Lambda(A, B)$ and $L = L(A,B)$. This result follows directly from (2.12).

## 2.3 M Functions

Consider first a scattering process involving one spin-$\frac{1}{2}$ particle in the initial state and one spin-$\frac{1}{2}$ particle in the final state, and an arbitrary number of spinless particles. Let $p \equiv (p_a, t_a; p_b, t_b; , p_c, t_c; \ldots; p_d, t_d)$, where $p_a$ is the mathematical momentum-energy of the final spin-$\frac{1}{2}$ particle, $p_b$

is the mathematical momentum-energy of the initial spin-½ particle, and $p_c, \ldots, p_d$ are the mathematical momentum-energy vectors of the spinless particles. The mathematical momentum-energy vectors are equal to plus or minus the physical momentum-energy vectors for final and initial particles respectively. Thus $p_a = m_a v_a$ and $p_b = - m_b v_b$, where $v_a^0 > 0$ and $v_b^0 > 0$.

The $t_j$ are the mathematical type labels. They are related to the physical-type labels $t_j^{phys}$ by the relation $t_j = t_j^{phys}/\text{sign } p_j^0$, where $t_j$ and $-t_j$ label relative antiparticles. These type variables are sometimes suppressed.

According to quantum theory the probability for a scattering specified by $(p, s_a, s_b)$ is proportional to

$$\frac{1}{2} \text{Tr } \tilde{P}^r(s_a) \, S(p) \, \tilde{P}^r(s_b) \, S^\dagger(p) , \qquad (2.26)$$

where $S(p)$ is the $S$ matrix. This can be written equivalently as

$$\frac{1}{2} \text{Tr } \tilde{P}(s_a, v_a) \, M(p) \, \tilde{P}(s_b, v_b) \, M^\dagger(p) , \qquad (2.27)$$

where, as in §2.2,

$$\tilde{P}(s_a, v_a) = \sqrt{v_a \cdot \tilde{\sigma}} \, \tilde{P}^r(s_a) \, \sqrt{v_a \cdot \tilde{\sigma}} \qquad (2.28a)$$

$$\tilde{P}(s_b, v_b) = \sqrt{v_b \cdot \tilde{\sigma}} \, \tilde{P}^r(s_b) \, \sqrt{v_b \cdot \tilde{\sigma}} , \qquad (2.28b)$$

and

$$M(p) = \sqrt{v_a \cdot \sigma} \, S(p) \, \sqrt{v_b \cdot \sigma} \qquad (2.29a)$$

$$M^\dagger(p) = \sqrt{v_b \cdot \sigma} \, S^\dagger(p) \, \sqrt{v_a \cdot \sigma} . \qquad (2.29b)$$

The physical probability is assumed to be Lorentz invariant. This

physical invariance ensures that if the spin indices of $M(p)$ and $M^\dagger(p)$ are assigned spin-index type according to the rules

$$M(p) \to M_{\alpha\dot{\beta}}(p) \qquad (2.30a)$$

and

$$M^\dagger(p) \to M_{\alpha\dot{\beta}}^\dagger(p), \qquad (2.30b)$$

then the spinor functions $M(p)$ and $M^\dagger(p)$ are Lorentz invariant: for all proper $(\det L = 1)$ real Lorentz transformations

$$\Lambda M(L^{-1}(p)) = M(p) \qquad (2.31a)$$

and

$$\Lambda M^\dagger(L^{-1}(p)) = M(p) , \qquad (2.31b)$$

with

$$L^{-1}(p) \equiv (L^{-1}p_a, L^{-1}p_b, L^{-1}p_c, \ldots, L^{-1}p_d). \qquad (2.31c)$$

These invariance properties entail that if $m^\mu(p)$ and $m^{\dagger\mu}(p)$ are defined by

$$M(p) \equiv m^\mu(p) \, \sigma_\mu \equiv m(p) \cdot \sigma \qquad (2.32a)$$

and

$$M^\dagger(p) \equiv m^{\dagger\mu}(p)\sigma_\mu \equiv m^\dagger(p) \cdot \sigma \qquad (2.32b)$$

then the quantities $m^\mu(p)$ and $m^{\dagger\mu}(p)$ are vector functions of the set of vectors $p$:

$$m^\mu(L(p)) = L^\mu_{\ \nu} m^\nu(p) = (Lm(p)) \qquad (2.33a)$$

$$m^{\dagger\mu}(L(p)) = L^\mu_{\ \nu} m^{\dagger\nu}(p) = (Lm^\dagger(p))^\mu . \qquad (2.33b)$$

Consequently, by virtue of (2.12), the spinor functions

$$\tilde{M}(p) \equiv m^\mu(p)\tilde{\sigma}_\mu = m(p)\cdot\tilde{\sigma} \tag{2.34a}$$

and

$$\tilde{M}^\dagger(p) \equiv m^{\dagger\mu}(p)\tilde{\sigma}_\mu = m^\dagger(p)\cdot\tilde{\sigma} \tag{2.34b}$$

are also Lorentz invariant spinor functions:

$$\Lambda\tilde{M}(L^{-1}(p)) = \tilde{M}(p) \tag{2.35a}$$

and

$$\Lambda\tilde{M}^\dagger(L^{-1}(p)) = \tilde{M}(p) . \tag{2.35b}$$

These simple transformation properties do not hold for the S-matrix $S(p)$.

The foregoing discussion can be immediately extended to processes in which there are $n$ initial spin $\frac{1}{2}$ particles, $n$ final spin $\frac{1}{2}$ particles, and $n'$ spinless particles. In this case the M function can be written in the form (with type labels suppressed)

$$M(p_{a1},\alpha_1; p_{b1},\dot{\beta}_1; p_{a2}, \alpha_2; p_{b2}, \dot{\beta}_2; \cdots; p_{an}, \alpha_n; p_{bn}, \dot{\beta}_n; p_1,\ldots,p_{n'})$$

$$= m^{\mu_1\mu_2\cdots\mu_n}(p) \times \prod_{i=1}^{n} \sigma_{\mu_i\alpha_i\beta_i} \tag{2.36}$$

where $m^{\mu_1\cdots\mu_n}(p)$ is a tensor function of the vectors

$p = (p_{a1}, p_{b1}, \cdots, p_{an}, p_{bn}, p_1, \ldots, p_{n'})$:

$$m^{\mu_1\cdots\mu_n}(L(p)) = \left(\prod_{i=1}^{n} L^{\mu_i}{}_{\nu_i}\right) m^{\nu_1\cdots\nu_n}(p). \tag{2.37}$$

The way in which the $n$ initial spin-$\frac{1}{2}$ particles are associated with the $n$ final spin-$\frac{1}{2}$ particles is immaterial: (2.37) holds in any case.

## 2.4 Parity

Let $S(p)$ be written as $S(p) = S_+(p) + S_-(p)$, $\qquad$ (2.38)

where

$$S_\pm(p) = \pm S_\pm(\tilde{p}) . \tag{2.39}$$

Here

$$\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_N) ,$$

$$(\tilde{p}_i^\mu) = (p_L^o, -\vec{p}_i), \tag{2.40a}$$

and

$$(p_i^\mu) = (p_i^o, \vec{p}_i) . \tag{2.40b}$$

Let an intrinsic parity $\varepsilon_j$ be assigned to each particle $j$, and define the parity operator $\mathscr{P}$ by

$$\mathscr{P}(S(p)) = \prod_{j=1}^{N} \varepsilon_j S(\tilde{p}) . \tag{2.41}$$

The product of $\varepsilon_i$'s for allowed processes must be $+1$ or $-1$. Invariance under parity is then expressed by the equation

$$\mathscr{P}(S(p)) = S(p). \tag{2.42}$$

If this equation is satisfied then $S_\varepsilon$ defined in (2.38) must be zero unless $\varepsilon = \prod\limits_{j=1}^{N} \varepsilon_j$ .

Consider a process in which $n$ initial spin-½ particles, $i = 1, \ldots, n$, are scattered into $n$ final spin-½ particles. Let $p_{ai}$ and $p_{bi}$ denote the final and initial mathematical momentum-energies of the $i^{th}$ particle. Let $(p_1, \ldots, p_{n'})$ denote the momenta of $n'$ spinless particles that also participate in the reaction. Then as already mentioned, the $M$ matrix can be written

$$M(p_{a1}, t_{a1}; p_{b1}, t_{b1}; \cdots ; p_{an}, t_{an}; p_{bn}, t_{bn}; p_1, t_1; \ldots; p_{n'}, t_{n'})$$

$$= m^{\mu_1 \cdots \mu_n}(p, t) \prod_{i=1}^{n} \sigma_{\mu_i}^{(i)} , \tag{2.43}$$

where the matrix elements of $\sigma_{\mu_i}^{(i)}$ are $\sigma_{\mu_i \alpha_i \dot{\beta}_i}$ . The connection of $M(p)$ to $S(p)$ can be represented by the equation

$$M(p) = (\prod_i \sqrt{v_{ai} \cdot \sigma^{(1)}}) \, S(p) \, (\prod_i \sqrt{v_{bi} \cdot \sigma^{(1)}})$$

$$\equiv \sqrt{V_a \cdot \sigma} \, S(p) \, \sqrt{V_b \cdot \sigma} \ . \tag{2.44}$$

Define now

$$M_\pm(p) = (\prod_i \sqrt{v_{ai} \cdot \sigma^{(1)}}) \, S_\pm(p) \, (\prod_i \sqrt{v_{bi} \cdot \sigma^{(1)}}) . \tag{2.45}$$

Then

$$M_\pm(\tilde{p}) = \left(\prod_i \sqrt{\tilde{v}_{ai} \cdot \sigma^{(1)}}\right) S_\pm(\tilde{p}) \left(\prod_i \sqrt{\tilde{v}_{bi} \cdot \sigma^{(1)}}\right)$$

$$= \pm \left(\prod_i \sqrt{\tilde{v}_{ai} \cdot \sigma^{(1)}}\right) S_\pm(p) \left(\prod_i \sqrt{\tilde{v}_{bi} \cdot \sigma^{(1)}}\right)$$

$$= \pm \, (\prod_i v_{ai} \cdot \tilde{\sigma}^{(i)}) M_\pm(p) (\prod_i v_{bi} \cdot \tilde{\sigma}^{(i)}). \tag{2.46}$$

This equation can be inverted to give

$$M_\pm(p) = \pm(\prod_i v_{ai} \cdot \sigma^{(i)}) \, M_\pm(\tilde{p}) (\prod_i v_{bi} \cdot \sigma^{(i)}). \tag{2.47}$$

The parity transformation applied to the M functions is defined to be

$$\mathscr{P}(M(p)) \equiv (\prod_{j=1}^{N} \varepsilon_j) (\prod_i v_{ai} \cdot \sigma^{(i)}) \, M(\tilde{p}) \, (\prod_i v_{bi} \cdot \sigma^{(i)}). \tag{2.48}$$

Then (2.48) and (2.47) ensure that the condition

$$\mathscr{P}(M(p)) = M(p) \tag{2.49}$$

is equivalent to the condition that $M_\varepsilon$ be zero unless $\varepsilon = \prod\limits_{j=1}^{N} \varepsilon_j$, which is equivalent to the parity invariance condition $\mathscr{P}(S(p)) = S(p)$.

For $n$ distinguishable spin-½ particles the no-scattering part of the $S$ matrix has the form

$$S_0(p) = \prod_{i=1}^{n} \left[ (\sigma_0^{(i)})(2\pi)^3 \delta^3(p_{ai} + p_{bi}) \, 2\omega_i \right]. \tag{2.50}$$

The corresponding $M$ function is

$$M_0(p) = \prod_{i=1}^{n} [v_i \cdot \sigma^{(i)} (2\pi)^3 \delta^3 (p_{ai} + p_{bi}) 2\omega_i] \quad . \qquad (2.51)$$

In order that this no-scattering part be invariant under parity (for each particle $i$ separately) we must take $\epsilon_{ai} \epsilon_{bi} = 1$ for all $i$. But then (2.48) gives

$$\mathscr{P}(p_a \cdot \sigma) = (-p_b \cdot \sigma) \quad . \qquad (2.52)$$

This relationship, which stems from the condition that the no-scattering part be nonzero, is used later.

## 2.5 Crossing

Analysis of the pole singularity[11] shows that the analytic continuation of $M(p)$ along an appropriate path from an original region where $p_{ai}^0 > 0$ to a region where $p_{ai}^0 < 0$ gives the function that describes a process in which the final particle of type $t_{ai}$ is replaced by an initial particle of type $-t_{ai}$, i. e., by the antiparticle of the original particle of type $t_{ai}$. If the final particle $t_{ai}$ carries $q$ units of any conserved quantity out of the reaction then the antiparticle $-t_{ai}$ must carry $-q$ units into the reaction. This holds both for the total momentum-energy $p_{ai}$, for the components of spin, and for any quantity that is conserved by virtue of invariance under a $p$-independent transformation property. Consequently, the mathematical momentum-energy vector $p_{ai}$, the mathematical spin vector $s_{ai}$, and the mathematical type label $t_{ai}$ are equal, after the continuation, to minus their physical values:

$$p_{ai} = p_{ai}^{phys} / \text{sign } p_{ai}^0 \qquad (2.53a)$$

$$s_{ai} = s_{ai}^{phys} / \text{sign } p_{ai}^0 \qquad (2.53b)$$

$$t_{ai} = t_{ai}^{phys} / \text{sign } p_{ai}^0 \quad . \qquad (2.53c)$$

A similar argument gives

$$p_{bi} = p_{bi}^{phys} / \text{sign } p_{bi}^0 \qquad (2.53d)$$

$$s_{bi} = - s_{bi}^{phys} / \text{sign } p_{bi}^0 \qquad (2.53e)$$

$$t_{bi} = t_{bi}^{phys} / \text{sign } p_{bi}^0 . \qquad (2.53f)$$

The minus sign in (2.53e) arises from the fact that $s_{bi}$ characterizes the physical spin of the initial particle $bi$, not minus the physical spin. The $p_{bi}$ and $t_{bi}$ were defined originally to be minus the physical momentum-energy vector and minus the physical particle-type of the incoming particle (bi), and hence the equations for these are the same as those for $p_{ai}$ and $t_{ai}$.

The quantitites occurring in the transition probability formula

$$\text{Tr } \frac{1}{2} (v_a + s_a) \cdot \tilde{\sigma} \, M(p_a, t_a; p_b, t_b)$$

$$\times \frac{1}{2} (v_b + s_b) \cdot \tilde{\sigma} \, M^\dagger(p_a, t_a; p_b, t_b) \qquad (2.54)$$

are to be interpreted with the aid of (2.53). Thus, for example, if

$p_a^o$ and $p_b^o$ are both positive then the $s_a$ and $s_b$ in (2.54) are $s_a^{phys}$ and $-s_b^{phys}$, respectively, and the particle types $t_a$ and $t_b$ and $t_a^{phys}$ are $t_b^{phys}$. In this way we can use the same expression (2.54) in all the different channels.

The parity transformation $\mathscr{P}$ was defined to be

$$\mathscr{P}(M(p)) = \left(\prod_{j=1}^{N} \varepsilon_j\right)\left(\prod_{i=1}^{n} v_{ai}\cdot\sigma\right) M(\tilde{p})\left(\prod_{L=1}^{n} v_{bi}\cdot\sigma\right) . \qquad (2.55)$$

In the original (direct) channel $(p_{ai}^o > 0, p_{bi}^o < 0)$ the parity invariance equation $\mathscr{P}(M(p)) = M(p)$ can be written as

$$M(p) = \left(\prod_{j=1}^{N} \varepsilon_j\right)\left(\prod_{i=1}^{n} \frac{p_{ai}\cdot\sigma}{m_{ai}}\right) M(\tilde{p})\left(\prod_{i=1}^{n} \frac{-p_{bi}\cdot\sigma}{m_{bi}}\right)$$

$$= \left(\prod_{j=1}^{n'} \varepsilon_j\right)\left(\prod_{i=1}^{n} \frac{p_{ai}\cdot\sigma}{m_{ai}}\right) M(\tilde{p})\left(\prod_{i=1}^{n} \frac{-p_{bi}\cdot\sigma}{m_{bi}}\right) , \qquad (2.56)$$

where use has been made of the direct-channel result $\varepsilon_{ai}\varepsilon_{bi} = 1$, derived from forward scattering. (See (2.51)).

Analytic continuation to the crossed channel avoids all singularities of $M(p)$ and $M(\tilde{p})$. Thus equation (2.56) must hold in all channels, with the factor $\prod_{j=1}^{n'} \varepsilon_j$ from the spinless particles defined as in the original direct channel. This equation gives

$$M(p) = \left(\prod_{j=1}^{n'} \varepsilon_j\right)\left(\prod_{i=1}^{n} \mathrm{sign}\ p_{ai}^o\right)\left(\prod_{i=1}^{n} \mathrm{sign}\ p_{bi}^o\right)$$

$$\times \left(\prod_{i=1}^{n} v_{ai}\cdot\sigma\right) M(\tilde{p})\left(\prod_{i=1}^{n} v_{bi}\cdot\sigma\right) \qquad (2.57)$$

It will be shown presently that the parity transformation is defined in all channels by (2.55). Thus one can conclude that the $\varepsilon_j$ for the spinless particles is channel independent and that

$$\varepsilon_{ai}\varepsilon_{bi} = -\ \mathrm{sign}\ p_{ai}^o\ \mathrm{sign}\ p_{bi}^o\ . \qquad (2.58)$$

This means, in particular, that the intrinsic parity of each spin-½ particle must reverse under continuation to a crossed channel and that the intrinsic parity of a particle-antiparticle pair is $-(-1)^\ell$.

The product of the intrinsic parities of the particles of a parity conserving process is physically well defined: it is equal to the sign $\varepsilon$ in $S(\tilde{p}) = \varepsilon S(p)$, and hence to $(-1)^{\Sigma\ell_j}$. The argument leading to the equivalence of $\mathscr{P}(S(p) = S(p))$ to $\mathscr{P}(M(p)) = M(p)$, with $\mathscr{P}$ as defined in (2.55), was made explicitly in the direct channel. However, it holds equally well in all channels, provided the same factor $\prod_{j=1}^{N} \varepsilon_j$ occurs in both $\mathscr{P}(S(p))$ and $\mathscr{P}(M(p))$. Any extra sign or phase factor $e^{i\phi}$, that one might introduce into the connection between $S(p)$ and $M(p)$, in any given physical region, would be the same throughout that physical region and would drop out of (2.47), and hence not affect the argument that demonstrates the equivalence between $\mathscr{P}(S(p)) = S(p)$ and $\mathscr{P}(M(p)) = M(p)$, with $\mathscr{P}(M(p))$ defined as in (2.48) or (2.55). Thus this definition is applicable in all channels, and the result (2.58) on the intrinsic parities of spin ½ particles holds.[12]

## 2.6 Antiparticle Conjugation

Consider a process in which $p_{ai}^o$ and $p_{bi}^o$ are both positive, so that the two associated particles are both final particles.

Suppose that $t_{ai} = -t_{bi}$, so that these two final particles are relative antiparticles.

Consider now an original value of $(p_{ai}, p_{bi})$ and an analytic continuation that stays in the physical region of the process, but interchanges $p_{ai}$ and $p_{bi}$ leaving all other p's unchanged. Suppose we interchange also $s_a^{phys}$ and $s_b^{phys}$. Then the original process and the second one are physically the same except for the interchange $t_{ai} \leftrightarrow t_{bi}$, which is just $t_{ai} \leftrightarrow -t_{ai}$.

Suppose that the transition probabilities for these two processes were the same. Then the process would be invariant under the transformation $t_{ai} \leftrightarrow -t_{ai}$. Antiparticle conjugation invariance is invariance under the analogous change $t_{ai} \leftrightarrow -t_{ai}$ for all i.

If we keep only one particle-antiparticle pair, for notational simplicity, the antiparticle conjugation invariance condition described above is

$$Tr(v_a + s_a^{phys}) \cdot \tilde{\sigma} \ M(p_a, p_b) (v_b - s_b^{phys}) \cdot \tilde{\sigma} \ M^{\dagger}(p_a, p_b)$$

$$= Tr \ (v_a' + s_a'^{phys}) \tilde{\sigma} \ M(p_a', p_b')(v_b' - s_b'^{phys}) \cdot \tilde{\sigma} \ M^{\dagger}(p_a', p_b'), \quad (2.59)$$

where (2.53) and (2.54) are used, and

$$p_a' = p_b \qquad p_a'^0 > 0$$

$$p_b' = p_a \qquad p_b'^0 > 0$$

$$s_a'^{phys} = s_b^{phys}$$

$$s_b'^{phys} = s_a^{phys} . \qquad (2.60)$$

To see the consequences of this condition define

$$\mathscr{C} M(p_a, t; p_b, -t) = u_a \cdot \sigma \ \tilde{M}(p_b, t; p_a -t) u_b \cdot \sigma \qquad (2.61)$$

where $u_a = p_a/m_a$ and $u_b = p_b/m_b$. Define also

$$M_{(\pm)} = \frac{1}{2} \ (1 \pm \mathscr{C}) \ M. \qquad (2.62)$$

Then $M = M_{(+)} + M_{(-1)}$, and the property $(\mathscr{C})^2 = 1$ gives

$$\mathscr{C}M = M_{(+)} - M_{(-)} . \qquad (2.63)$$

Hence if $M = M_{(+)}$ or $M_{(-)}$ then $M = \pm \mathscr{C}M$.

Insertion of this condition $M = \pm \mathscr{C} M$ into the LH side of (2.59) gives

$$Tr \ (v_a + s_a^{phys}) \cdot \tilde{\sigma} \ u_a \cdot \sigma \quad \tilde{M}(p_b, p_a) u_b \cdot \sigma$$

$$\times (v_b - s_b^{phys}) \cdot \tilde{\sigma} \ u_b \cdot \sigma \ \tilde{M}^{\dagger} (p_b, p_a) u_a \cdot \sigma$$

$$= Tr \ (v_a - s_a^{phys}) \cdot \sigma \ \tilde{M} \ (p_b, p_a)$$

$$\times (v_b + s_b^{phys}) \cdot \sigma \ \tilde{M}^{\dagger} (p_b, p_a)$$

$$= Tr \ (v_b + s_b^{phys}) \cdot \tilde{\sigma} \ M(p_b, p_a)(v_a - s_a^{phys}) \cdot \tilde{\sigma}$$

$$\times M^{\dagger}(p_b, p_a) , \qquad (2.64)$$

where in the second line the relations

$$u_a \cdot \sigma \; u_a \cdot \tilde{\sigma} = 1 \qquad (2.65a)$$

and

$$s_a \cdot \tilde{\sigma} \; u_a \cdot \sigma = {}^- u_a \cdot \tilde{\sigma} \; s_a \cdot \sigma \; \text{for} \; s_a \cdot u_a = 0 \qquad (2.65b)$$

are used, and in the last line the equations

$$\sigma = C^{-1} \; \tilde{\sigma}^{Tr} C \qquad (2.66a)$$

and

$$\tilde{\sigma} = C^{-1} \; \sigma^{Tr} C \qquad (2.66b)$$

are used. Comparison of (2.64) to the RH side of (2.59), with the substitutions (2.60) made, shows that the condition $M = \mathscr{C} M$ implies antiparticle conjugation invariance.

Notice that

$$\mathscr{C} p_a \cdot \sigma = p_a \cdot \sigma \qquad (2.67a)$$

and

$$\mathscr{C} p_b \cdot \sigma = p_b \cdot \sigma. \qquad (2.67b)$$

Thus both $p_a \cdot \sigma$ and $p_b \cdot \sigma$, and any superposition of them, is invariant under $\mathscr{C}$.

## 2.7 CPT Invariance

The physical transformation corresponding to CPT is
$$p_j^{phys} \to p_j^{phys} , \; s_j^{phys} \to - s_j^{phys} , \; t_j^{phys} \to - t_j^{phys} , \; \text{In} \leftrightarrow \text{Out.}$$
$$(2.68a)$$

The corresponding mathematical transformation is, by virtue of (2.53),

$$p_j \to - p_j, \; s_j \to s_j, \; t_j \to t_j. \qquad (2.68b)$$

Thus CPT invariance is equivalent to invariance of transition probabilities under the transformation $p_j \to - p_j$ (all j).

Any Lorentz invariant spinor function $M(p)$ is invariant, up to a sign, under the transformation $p_j \to - p_j$ (all j). For the Lorentz invariance condition

$$\Lambda M(L^{-1}(p)) = M(p) \qquad (2.69)$$

applied for the case $A = 1$, $B = -1$ gives, by virtue of (2.3) and (2.9),

$$M(-p) = (-1)^{N_d} M(p), \qquad (2.70)$$

where $N_d$ is the number of dotted spinor indices (I mean here dotted two-valued spinor indices: Dotted (undotted) spinor indices for spin $\frac{n}{2}$ particles can be constructed trivially by combining $n + 2m$ dotted (undotted) two-valued spinor indices by means of the usual Clebsch-Gordan coeffieients. Thus a dotted spin $\frac{n}{2}$ spinor index contributes a term $n$ to $N_d$)

The matrix $B = -1$ can be continuously connected to $B = 1$ by the matrix

$$B(\psi) = \begin{pmatrix} e^{i\pi\psi} & 0 \\ 0 & e^{-i\pi\psi} \end{pmatrix} \qquad (2.71)$$

which satisfies $B(0) = 1$ and $B(1) = -1$.

Since all Lorentz invariants are invariant under all real and complex Lorentz transformations the transformation $L(A,B) = L(1, B(\psi))$, with $0 \leqslant \psi \leqslant 1$, must generate complex values of the $p_j$, since no real mass-shell vector $p_j(\psi)$ can interpolate $p_j \to -p_j$.

The matrices $M(p)$ and $\overset{+}{M}(p)$ have been assigned the transformation properties indicated by the indices $M_{\alpha\dot\beta}(p)$ and $M^{+}_{\alpha\dot\beta}(p)$. For real $p$ the matrices $M(p)$ and $M^{\dagger}(p)$ are related by hermitian conjugation:

$$M^{+}_{\alpha\dot\beta}(p) = (M_{\beta\dot\alpha}(p))^{*}. \qquad (2.72a)$$

Thus if $M$ is transformed by a real Lorentz transformation to $AMB$ then $M^{*}$ is transformed to $A^{*}M^{*}B^{*}$ and $M^{\dagger}$ is transformed to $B^{\dagger}M^{\dagger}A^{\dagger} = AM^{\dagger}B$, as indicated by the indices on $M^{\dagger}_{\alpha\dot\beta}$ .

For complex Lorentz transformations the condition $A = B^{\dagger}$ does not hold. However, (2.72a) is then inappropriate: the appropriate definition is

$$M^{\dagger}_{\alpha\dot\beta}(p) = M^{*}_{\underset{\dot\beta\alpha}{}}(p) \equiv (M_{\beta\dot\alpha}(p^{*}))^{*}. \qquad (2.72b)$$

This quantity is an analytic function of $p$, whereas the function on the RH side of (2.72a) is an analytic function of $p^{*}$. The function $M^{\dagger}$ defined in (2.72b) will continue to satisfy the Lorentz invariance condition

$$\left( \prod_{i=1}^{n} A^{(i)} \right) M^{\dagger}(L^{-1}(A,B)(p)) \left( \prod_{i=1}^{n} B^{(i)} \right)$$
$$= M^{\dagger}(p) \qquad (2.73)$$

for complex Lorentz transformations.[13] Thus in the formula for probabilities the factor $(-1)^{N_d}$ from (2.70) will be cancelled by the same factor $(-1)^{N_d}$ from

$$M^{+}(-p) = (-1)^{N_d} M^{+}(p) . \qquad (2.74)$$

Hence probabilities will be invariant under CPT.

## 2.8 Statistics

The order of writing the variables is important. If the variables in the set of arguments $p = (p_1, t_1; p_2, t_2; \ldots ; p_n, t_n)$ is such that all variables referring to initial particles stand to the right of all variables referring to final particles then one may write $p = (p_{fin}; p_{in})$. By convention

$$S(p_{fin}; p_{in}) \equiv \langle p_{fin} \mid S \mid \bar{p}_{in} \rangle, \qquad (2.75)$$

where $\bar{p}_{in}$ is obtained from $p_{in}$ by reversing the signs of all energy vectors $p_j$ and all type variables $t_j$, and reversing the order of the variables. Thus if

$$p_{in} = (p_m, t_m; \ldots ; p_n, t_n) \qquad (2.76a)$$

then

$$\bar{p}_{in} = (-p_n, -t_n; \ldots ; -p_m, -t_m). \qquad (2.76b)$$

The diagram representing $\langle p_{fin} \mid S \mid \bar{p}_{in} \rangle$ is generally drawn by ordering the lines from top to bottom in the sequence in

which the corresponding arguments of $p_{fin}$ and $\bar{p}_{in}$ appear. The lines corresponding to $\bar{p}_{in}$ are on the right-hand side; those corresponding to $p_{fin}$ are on the left-hand side. The variables in (2.50) are in the order $(p_{1a}, p_{2a}, \cdots p_{2b}, p_{1b})$, so that each particle line goes straight through, without a change in order.

The functions $S(p)$ and $M(p)$ are assumed to be anti symmetric under the interchange of any two spin-½ particle variables $(p_i, t_i)$ and $(p_j, t_j)$. Analytic continuation $p_{ai} \to p_{bi}$ in (2.51) changes the sign of (2.51). This sign change is cancelled by the change of the order of variables required to bring the variables back into the form $(p_{fin}; p_{in})$. Thus (2.50) and (2.51) hold in all channels, for $p = (p_{fin}; p_{in})$, with the corresponding variables of $p_{fin}$ and $\bar{p}_{in}$ occurring in the same order.

With these conventions the relationship (2.45) between $M(p)$ and $S(p)$ holds in all channels.

Combinatoric factors $1/n!$ are discussed in Appendix A of Ref. 14.

### 3. MESONS

#### 3.1 The Zero-Entropy Amplitudes

The basic building blocks of the topological expansion are the zero-entropy amplitudes. In the meson sector each zero-entropy amplitude is represented by a simple quark diagram D of the kind shown in Fig. 1, or by the equivalent quark graph G also shown there.
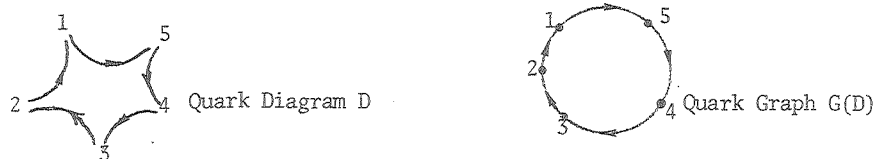


Figure 1 A zero-entropy quark diagram D and the equivalent quark graph $G = G(D)$.

The quark diagram D is converted to the equivalent quark graph $G = G(D)$ by simply connecting to a vertex the ends of the two quark lines at each opening of D. Thus each vertex of a meson quark graph G corresponds, at some level of approximation, to an initial or final particle of a scattering process. The zero-entropy amplitude corresponding to a process with n particles is represented, therefore, by a directed circular graph with n vertices. The n directed edges that connect these vertices all run in the same direction, as illustrated in Fig. 1.

The quark graphs are not abstract graphs, but are graphs placed on an oriented surface. The orientation of the boundary of the oriented circular disc bounded by the quark line is indicated by a second arrow, as shown in Figs. 2 and 3. The two graphs of Fig. 2 are equivalent to each other, and the two graphs of Fig. 3 are equivalent to each other. But those of Fig. 2 are not equivalent to those of Fig. 3.



Fig. 2. Two equivalent "ortho" graphs $G^O$



Fig. 3. Two equivalent "para" graphs $G^p$

The circular graphs in which the directions of all the quark lines agree with the direction of the boundary of the enclosed oriented disc , as in Fig. 2, are called "ortho" graphs. The circular graphs in which the directions of all the quark lines are opposite to the direction of the boundary of the oriented disc, as in Fig. 3, are called "para" graphs.

For each ortho or para graph $G$ there is a corresponding amplitude. If $G$ has $n$ vertices then this amplitude has a set $(\mu) = (\mu_1, \ldots, \mu_n)$ of $n$ vector indices. The amplitude corresponding to $G$ has the form

$$A_{(\mu)}(G, p) = F_{(\mu)}(G, p)f(G, p) , \qquad (3.1)$$

where $f(G, p)$ is a function of the scalar products of the mathematical momentum-energy vectors $p_j$ appearing in the set of arguments $p = (p_1, t_1; \ldots; p_n, t_n)$. For any ortho graph $G = G^o$ the function $F_{(\mu)}(G^o, p)$ is given explicitly by

$$F_{(\mu)}(G^o, p) = - \prod_{i=1}^{n} (2m_i^2)^{-\frac{1}{2}}$$

$$\times \ \mathrm{Tr}\ \tilde{\sigma}_{\mu_1} p_1 \cdot \sigma \ \tilde{\sigma}_{\mu_2} p_2 \cdot \sigma \cdots \tilde{\sigma}_{\mu_n} p_n \cdot \sigma \qquad (3.2)$$

This factor $F_{(\mu)}(G^o, p)$ is minus the trace of a matrix formed from right to left by following the sense of the quark arrows in $G^o$ and replacing each vertex $i$ by $\tilde{\sigma}_{\mu_i}/\sqrt{2}$ and each edge by the ortho quark "propagator" $p_{ai} \cdot \sigma / m_{ai} \equiv u_{ai} \cdot \sigma$ , where $p_{ai}$ is the mathematical momentum-energy vector associated with

the vertex that lies on the leading end of that quark edge.

If $G^p$ is the para graph obtained from $G^o$ by reversing the orientation of the disc then

$$A_{(\mu)}(G^p, p) = \mathscr{P} A_{(\mu)}(G^o, p)). \qquad (3.3a)$$

Thus the function $A(G, p)$ is invariant under the parity operation, in the sense that if $\mathscr{P} G^o = G^p$ and $\mathscr{P} G^p = G^o$ then

$$\mathscr{P} A_{(\mu)}(\mathscr{P} G, p) = A_{(\mu)}(G, p). \qquad (3.b)$$

The action of $\mathscr{P}$ on any $A$ is given by (2.48). Thus

$$\mathscr{P} f(G, p) = f(G, \tilde{p}), \qquad (3.4a)$$

and, by virtue of (2.52),

$$F_{(\mu)}(G^p, p) = - \prod_{i=1}^{n} (2m_i^2)^{-\frac{1}{2}}$$

$$\times \ \mathrm{Tr}(-p_1 \cdot \sigma) \ \tilde{\sigma}_{\mu_1} (-p_2 \cdot \sigma) \ \tilde{\sigma}_{\mu_2} \cdots (-p_n \cdot \sigma) \ \tilde{\sigma}_{\mu_n} \quad (3.4)$$

This spinor part of the para amplitude is minus the trace of a matrix formed from right to left by following the sense of the quark arrows in $G^p$ and replacing each vertex $i$ of $G^p$ by $\tilde{\sigma}_{\mu_i}/\sqrt{2}$ and each edge by the para quark "propagator" $- p_{bi} \cdot \sigma / m_{bi} \equiv - u_{bi} \cdot \sigma$, where $p_{bi}$ is the mathematical momentum-energy

vector associated with the vertex that lies on the <u>trailing</u> end of the quark edge.

Notice that in both the ortho and para cases the orientation of the disc points from each edge to the vertex whose momentum appears in the propagator corresponding to that edge.

Each vertex $i$ is associated with a spin four-vector $s_i$. For a vector particle $s_i \cdot p_i = 0$ and $s_i^2 = -1$. For a pseudo scalar particle $s_i = v_i$, and $s_i^2 = 1$. The vector $s_i$ is the "wave function" of particle $i$ in spin space. The ortho and para amplitudes themselves are therefore

$$A(G^o, p, s) = - \left( \frac{1}{\sqrt{2}} \right)^n Tr(s_1 \cdot \tilde{\sigma} \ u_1 \cdot \sigma \ \dots \ s_n \cdot \tilde{\sigma} \ u_n \cdot \sigma)$$

$$\times f(G^o, p) \qquad (3.5a)$$

and

$$A(G^p, p, s) = - \left( \frac{-1}{\sqrt{2}} \right)^n Tr(u_1 \cdot \sigma \ s_1 \cdot \tilde{\sigma} \ \dots \ u_n \cdot \sigma \ s_n \cdot \tilde{\sigma})$$

$$\times f(G^p, p) . \qquad (3.5b)$$

## 3.2 Parity

Let $G^o$ and $G^p$ be ortho and para graphs related by disc reversal. Since $A(G^o)$ and $A(G^p)$ are related by $A(G^p) = \mathscr{P} A(G^o)$, the sum $A(G^p) + A(G^o)$ is invariant under parity. To see this explicitly use

$$u_i \cdot \sigma \ s_i \cdot \tilde{\sigma} = \pm s_i \cdot \sigma \ u_i \cdot \tilde{\sigma}$$

$$\begin{cases} + \text{ for spin} & 0 \\ - \text{ for spin} & 1 \end{cases} \qquad (3.6)$$

to obtain

$$A(G^p, p, s) = - \left( \frac{-1}{\sqrt{2}} \right)^n (-1)^{\text{no. of spin 1's}}$$

$$\times Tr \ s_1 \cdot \sigma \ u_1 \cdot \tilde{\sigma} \ \dots \ s_n \cdot \sigma \ u_n \cdot \tilde{\sigma}$$

$$\times f^p(p)$$

$$= - \left( \frac{1}{\sqrt{2}} \right)^n (-1)^{\text{no. of spin 0's}}$$

$$\times Tr \ s_1 \cdot \sigma \ u_1 \cdot \tilde{\sigma} \ \dots \ s_n \cdot \sigma \ u_n \cdot \tilde{\sigma}$$

$$\times f^p(p) , \qquad (3.7)$$

where $f^p(p) \equiv f(G^p, p)$ and $f^o(p) = f(G^o, p)$.

Any trace $Tr \ a_1 \cdot \tilde{\sigma} \ a_2 \cdot \sigma \ a_3 \cdot \tilde{\sigma} \dots a_{2n} \cdot \sigma$ is a sum of a scalar part that is unchanged by $a_i \to \tilde{a}_i$ and a pseudoscalar part that changes sign. Since $f^p(p) = f^o(\tilde{p}) = f^o(p)$ the equations (3.5a) and (3.7) imply (with $A^o \equiv A(G^o)$, $A^p \equiv A(G^p)$) that

$$A^o + A^p = 2 \times \text{scalar part of } A^o \qquad (3.8a)$$

$$\text{if no. of spin zero's is even}$$

$$A^o + A^p = 2 \times \text{pseudoscalar part of } A^o$$

$$\text{if no. of spin zero's is odd.} \qquad (3.8b)$$

This means that $A^o + A^p$ conserves parity, provided the spin-zero particles are identified as pseudo scalar particles and the spin-one particles are identified as vector particles.

## 3.3 Antiparticle Conjugation

The ortho and para propagators are $(p_{ai} \cdot \sigma)/m_{ai}$ and $(-p_{bi} \cdot \sigma)/m_{bi}$, respectively. According to (2.67) these forms are invariant under the antiparticle conjugation operation $\mathscr{C}$. This result suggests that the ortho and para amplitudes should be separately invariant under antiparticle conjugation. This invariance would, in fact, be strictly implied if the quarks could be considered separate entities, each with its own initial and final momenta $p_{bi}$ and $p_{ai}$. It was the analytic continuation $p_{ai} \leftrightarrow p_{bi}$ of these momenta into each other that was the basis of the discussion of antiparticle conjugation in §2.6. In that context antiparticle conjugation was equivalent (up to a sign) to reversing the directions of all the quark arrows. This reversal was accomplished by an equivalent analytic continuation. In that continuation the vector p in the propagator $p \cdot \sigma/m$ continues to be the momentum associated with a fixed end of the quark line. Thus an ortho propagator is transformed into an ortho propagator, and para goes into para.

We therefore define antiparticle conjugation to be the operation of reversing the direction of each quark edge, with the ortho-para type left unchanged. Thus antiparticle conjugation interchanges the two graphs (a) and (b) of Fig. 4.
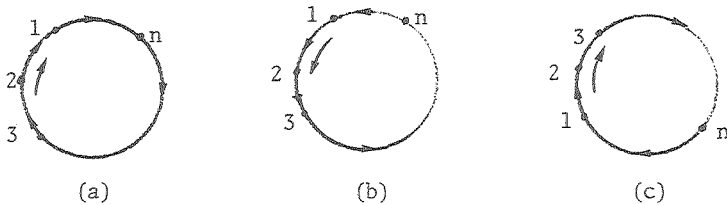


Fig. 4 Graphs (a) and (b) are related by antiparticle conjugation. Graph (c) is graph (b) turned over.

The scalar functions $f(p)$ are assumed to be unchanged by antiparticle conjugation. Thus the amplitudes associated with graphs (a) and (b) are

$$A_a^O = - \left( \frac{1}{\sqrt{2}} \right)^n (\mathrm{Tr}\ s_1 \cdot \tilde{\sigma}\ u_1 \cdot \sigma\ \ldots\ s_n \cdot \tilde{\sigma}\ u_n \cdot \sigma)\ f^O(p) \tag{3.9a}$$

and

$$A_b^O = - \left( \frac{1}{\sqrt{2}} \right)^n (\mathrm{Tr}\ s_n \cdot \tilde{\sigma}\ u_n \cdot \sigma\ \ldots\ s_1 \cdot \tilde{\sigma}\ n_1 \cdot \sigma)\ f^O(p) . \tag{3.9b}$$

Then use of (2.65) and (2.66) gives

$$A_b^O = (-1)^{\sum_i (\mathrm{Spin})_i} A_a^O \tag{3.10}$$

## 3.4 Isospin

Quark flavors have not yet been discussed. Introduction of the up and down quarks yields the $\pi, \rho, n,$ and $\omega$ mesons. To get the amplitude corresponding to a graph with these mesons as the external particles one includes for each vertex the isotopic spin factor $f_i$ defined in Fig. 5:
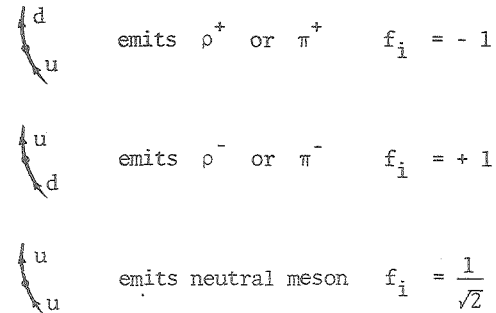


$$
\begin{array}{lll}
d\ / \ u & \text{emits } \rho^+ \text{ or } \pi^+ & f_i = -1 \\[6pt]
u\ / \ d & \text{emits } \rho^- \text{ or } \pi^- & f_i = +1 \\[6pt]
u\ / \ u & \text{emits neutral meson} & f_i = \dfrac{1}{\sqrt{2}}
\end{array}
$$

Figure continued

$$\begin{cases} d \\ u \end{cases} \text{emits neutral meson} \quad f_i = \begin{cases} -\dfrac{1}{\sqrt{2}} & \pi^0, \ \rho^0 \\[2mm] +\dfrac{1}{\sqrt{2}} & \eta^0, \ \omega^0 \end{cases}$$

Figure 5. The isotopic spin factors. The full zero-entropy amplitude for any process involving a set of $n$ of these mesons is the sum of the amplitudes corresponding to all the ways in which the particles of the reaction can be identified with the vertices of ortho and para graphs with $n$ vertices.

G-parity is conserved for the ortho and para amplitudes separately. To see this note that for each ortho (para) graph contributing to a process there is another one in which the $u$ and $d$ quarks are interchanged and the cyclic order of the particles is reversed. The two associated ortho graphs are related as the two graphs (a) and (c) of Fig.4 apart from flavor labels. Since (c) is equivalent to (b) one obtains the factor (3.10) together with the isospin factors $f_i$ shown in Fig.5. These factors $f_i$ combine to give factors for the graphs (a) and (c) that differ by the factor $\prod_i (-1)^{(\text{Isospin})_i}$. Thus the sum of the two contributions is

$$A_a^O + A_c^O = A_a^O \ (1 + (-1)^g) \tag{3.11a}$$

where

$$g = \sum_i (\text{Spin})_i + (\text{Isospin})_i \ . \tag{3.11b}$$

The factor $(-1)^g$ is G parity. Hence G parity is conserved for the ortho and para parts separately.

## 3.5 Products

The discontinuity formulas involve products of amplitudes represented by graphs of the kind shown in Fig. 6.
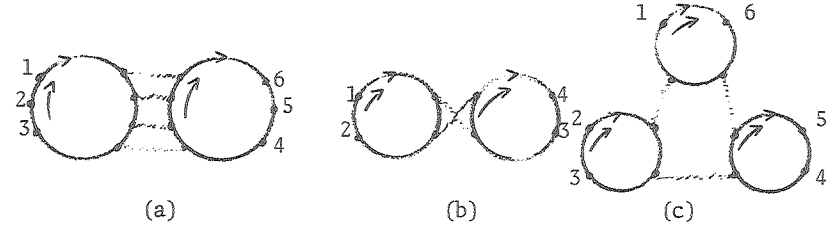


Figure 6. Diagrams representing products of amplitudes. The wiggly lines represent the intermediate mesons.

For each wiggly line there is a sum over the single pseudoscalar meson and the three vector mesons. When this sum is performed the spinor parts of these products are just the spinor parts of the functions associated with the diagrams of Fig. 7.
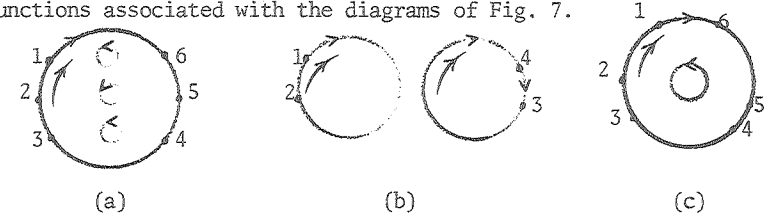


Figure 7. Alternative representation of the spinor parts of the products represented in Fig. 6. A circle with no vertices represents $-\text{Tr} \ 1 = -2$.

In other words, the spinor parts satisfy the diagrammatic equations of Fig. 8.



Fig. 8. Spinor Identities

To obtain this result, and also a more general one, let the four orthogonal vectors $s_i$ associated with particle $i$ be labelled by $e \in (0, 1, 2, 3)$, with $s_{ie}^u = v_i^u$ for $e = 0$. To get the correct normalization we return to the level of the $S$ matrix. Then the four amplitudes $S_e$ are defined by

$$S_e = \text{Tr} \, \frac{s_e^r \cdot \tilde{\sigma}}{\sqrt{2}} \, S$$

$$= \frac{1}{\sqrt{2}} s_e^r \cdot \tilde{\sigma}^{\dot{\alpha}\beta} \, S_{\beta\dot{\alpha}} \quad , \tag{3.12}$$

where the irrelevant indices on $S$ have been suppressed. The arguments of §2.3 then show that

$$S_e = \text{Tr} \, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} \, M. \tag{3.13}$$

Consider therefore a product of the form

$$\sum_{e=0}^{3} S_e S_e' = \sum_{e=0}^{3} (\text{Tr} \, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} \, M)(\text{Tr} \, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} \, M'). \tag{3.14}$$

To evaluate it introduce into the second trace the identity

$$s_e \cdot \tilde{\sigma} = v \cdot \tilde{\sigma} \, s^e \cdot \sigma \, v \cdot \tilde{\sigma} \quad , \tag{3.15}$$

where $v$ is the velocity $\pm p/m$ of the relevant particle and $s^0 = s_0$, $s^1 = -s_1$, $s^2 = -s_2$, $s^3 = -s_3$. (Each $s_e$ is a four vector).

Thus

$$\sum_{e=0}^{3} S_e S_e' =$$

$$\frac{1}{2} \sum_{e=0}^{3} (\text{Tr} \, M \, s_e \cdot \tilde{\sigma})(\text{Tr} \, s^e \cdot \sigma \, v \cdot \tilde{\sigma} \, M' \, v \cdot \tilde{\sigma}). \tag{3.16}$$

Use of

$$\sum_{e=0}^{3} (s_e)^\mu (s^e)^\nu = g^{\mu\nu} \tag{3.17}$$

gives

$$\sum_{e=0}^{3} S_e S_e' =$$

$$\frac{1}{2} (\text{Tr} \, M \, \tilde{\sigma}_\mu) \, g^{\mu\nu} (\text{Tr} \, \sigma_\nu v \cdot \tilde{\sigma} \, M' \, v \cdot \tilde{\sigma}).$$

Use of

$$\frac{1}{2} \tilde{\sigma}_\mu^{\dot{\alpha}\beta} \, g^{\mu\nu} \, \sigma_{\nu\gamma\dot{\delta}} = \delta^{\dot{\alpha}}_{\dot{\delta}} \, \delta^\beta_\gamma \tag{3.18}$$

gives

$$\sum_{e=0}^{3} S_e S_e' = \sum_{e=0}^{3} (\text{Tr} \, M \, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}})(\text{Tr} \, \frac{s_e \cdot \tilde{\sigma}}{\sqrt{2}} \, M')$$

$$= \text{Tr} \, M \, v \cdot \tilde{\sigma} \, M' \, v \cdot \tilde{\sigma} \, . \tag{3.19}$$

This result says that summing over all four exchanged particles is equivalent to running the quark lines straight through, with metric factors $v \cdot \tilde{\sigma}$ placed on each quark line that goes through, as indicated in Fig. 9.

Figure 9.  Diagrammatic representation of (3.19).

The results represented in Figs.  7  and  8 follow directly  from  the  result represented in Figure  9.

The two factors  $v \cdot \tilde{\sigma}$  are cancelled by two factors  $v \cdot \sigma$  and $-v \cdot \sigma$ .  The differing signs of these two factors comes from the differing signs of the vectors  $p = \pm mv$  in the two factors  M  and M':    discontinuity products always involve one initial particle and one  final particle.  The extra sign from  $-v \cdot \sigma$  supplies the over-all sign occurring in (3.5):  the change shown in Fig. 9 always changes by one the number of traces.  Thus each trace will appear (or disappear) with its minus sign.

### 3.6  Topological Classification

Each circular quark graph  G  corresponding to a zero-entropy ortho or para amplitude can be transformed by the rule illustrated in Fig. 10 into a particle graph  $g = g(G)$  with one internal vertex.
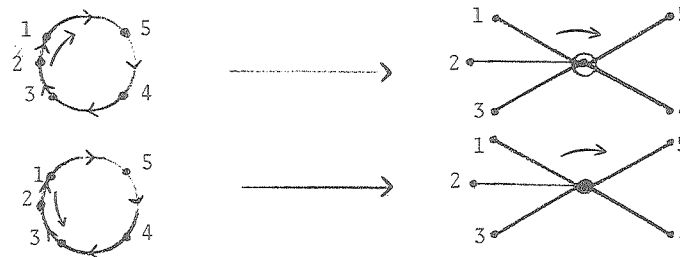


Figure 10. Transformation of circular ortho and para quark graphs  G into the corresponding basic particle graphs  g(G).

If  G  is a circular graph with  n  vertices then  g(G)  is a tree graph with  n  edges,  n  external vertices, and one internal vertex.  This internal vertex of  g(G)  is classified as ortho or para according to whether  G  is ortho or para.  These two kinds of internal vertices can be distinguished in the way illustrated in Fig. 10. The arrow near each internal vertex shows the direction of rotation of the quark line around that vertex.  These graphs  g  are called basic particle graphs.

A product of basic particle graphs $g_1, g_2, \ldots$ is formed by identifying certain pairs of the external vertices, as illustrated in Fig. 11.
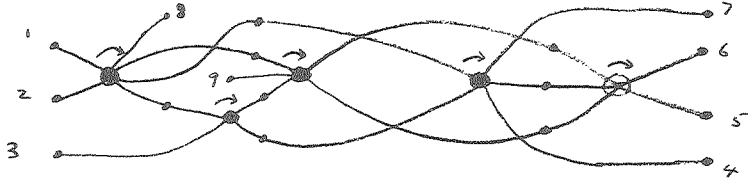


Figure 11. A product $g$ of 5 basic particle graphs $g_i$.

Each product graph $g$ has a well defined genus and boundary structure. These can be calculated by the Edmond's rule. One first draws all the orbits of $g$. An orbit of $g$ is a path in $g$ formed as follows: one picks any point $p$ on any edge of $g$ and a direction $d(p)$ at that point. Then one traces a path in $g$ by a moving point $p'$ that starts from $p$ in the direction $d(p)$. At each nontrivial vertex the moving point $p'$ shifts to the "next" line, with the order of the lines specified by the arrow that indicates the quark-line direction. The orbit is completed when the moving point $p'$ returns to the original point $p$ moving in the original direction $d(p)$.

Some of the orbits may pass through vertices that lie at the ends of single (external) edges. These vertices correspond to the "external particles" associated with the graph. An orbit that passes through at least one external-particle vertex is called a boundary. The boundary structure consists of the collection of boundaries, each identified by the sequence of external-particle vertices through which it passes. Each external-particle vertex appears on exactly one boundary. Graphs with only one boundary are called one-boundary graphs.

The number of different orbits of $g$--sometimes called faces

of $g$-- is denoted by $f(g)$. The numbers of edges and vertices of $g$ are denoted by $e(g)$ and $v(g)$, respectively. Then the genus of $g$-- sometimes called the handle number--is given by the Euler formula

$$h(g) = \frac{e(g) - v(g) - f(g) + 2c(g)}{2}, \qquad (3.20)$$

where $c(g)$ is the number of connected components of $g$. The graph of Fig. 11 has one orbit, which is the boundary $(8, 6, 5, 9, 7, 4, 3, 2, 1)$, and its genus is two.

The zero-genus one-boundary graphs are the planar graphs. They are the graphs that can be drawn on a plane with no lines crossing and all external vertices identified with a single point at infinity.

An important characteristic of a graph $g$ is its Betti number $\beta(g)$, which is the number of independent closed loops that can be drawn in the graph. Its value is given by

$$\beta(g) = e(g) - v(g) + c(g). \qquad (3.21)$$

Let the number of boundaries of $g$ be $b(g)$. The orbits that are not boundaries are called windows, and their number is $w(g) = f(g) - b(g)$. The most important topological characteristic of $g$ is the topological index

$$\gamma(g) = 2h(g) + b(g) - c(g)$$
$$= \beta(g) - w(g). \qquad (3.22)$$

This is the number of independent closed loops in g minus the number of windows. For connected graphs with at least one boundary the topological index $\gamma(g)$ is zero if and only if the graph g has zero-genus and exactly one boundary, i.e., if and only if g is planar.

This topological index $\gamma(g)$ enjoys the following "entropy" property: if $g_1 g_2$ is some connected product of two connected graphs $g_1$ and $g_2$ then

$$\gamma(g_1 g_2) \geqslant \gamma(g_1) + \gamma(g_2) - 1. \qquad (3.23)$$

To prove this let n be the number of vertices at which $g_1$ and $g_2$ are joined. Then (3.21) gives

$$\beta(g_1 g_2) = \beta(g_1) + \beta(g_2) - 1 + n. \qquad (3.24)$$

On the other hand,

$$w(g_1 g_2) = w(g_1) + w(g_2) + w'(g_1, g_2) \qquad (3.25)$$

where $w'(g_1, g_2)$ is the number of windows of $g_1 g_2$ that lie partly in $g_1$ and partly in $g_2$.

Each of these windows that lies partly in each subgraph must pass at least twice through the n junction points. And each junction point lies exactly twice on the set of orbits. Thus one has the inequality

$$w'(g_1, g_2) \leqslant n, \qquad (3.26)$$

which combines with (3.24) and (3.22) to give (3.23).

The entropy property (3.23) shows that the topological index $\gamma(g_1 g_2)$ of a product graph $g_1 g_2$ is greater than either component, provided one of them has $\gamma(g_i) \geqslant 1$. This means that the topological complexity, as measured by $\gamma(g)$, increases in general. The special case $\gamma(g_i) = 1$ allows the complexity to remain unchanged.

If one of the graphs has $\gamma(g_i) = 0$ then (3.23) would allow for a decrease in complexity. However, if $\gamma(g_1) = 0$ and the product $g_1 g_2$ is such that at least one external vertex of $g_1$ is an external vertex also of the product graph $g_1 g_2$ then the RH side of (3.26) can be replaced by $n - 1$, since then at least one boundary of $g_1 g_2$ must pass twice through the set of junction points, and $\gamma(g_1 g_2) \geqslant \gamma(g_1) + \gamma(g_2)$.

The graphs corresponding to physical-region singularities can always be constructed by taking successive products $g_1$, $g_2 g_1$, $g_3 g_2 g_1$, ... so that the final external particles of each newly added graph are also final external particles of the new product graph.[13] If the product graphs are built in this way then the topological index $\gamma(g)$ can never decrease.

The product graphs g are classified by their overall boundary structure and genus, and by their decomposition into ortho and para parts. This decomposition is made as follows: The trivial two-edge vertices at which two graphs are joined are called junction vertices. Each junction vertex that lies on a line joining an ortho vertex to a para vertex is cut. This cuts the graph into a set of graphs $g_i$ such that the internal vertices of each graph $g_i$ are all of the same kind, either ortho or para. Each of these graphs $g_i$ has a

boundary structure and genus. The complete topological classification

of the graph $g$ is given by specifying the boundary structure and

genus of each part $g_i$, and the set of pairs of external vertices of

the graphs $g_i$ that are equated to form the junction vertices of $g$.

These specifications determine the overall boundary structure

and genus of $g$ itself. However, these overall characteristics

are nevertheless included, redundantly, in the complete topological

specification of $g$.

Graphs $g$ that have the same topological specifications are

said to lie in the same topological class. The zero-entropy graphs

are the graphs $g$ with a single (ortho or para) part $g_i = g$ and

topological index $\gamma(g) = 0$. The simplest of these are the basic

graphs $g$ of the kind illustrated in Fig. 10.

The discussion of topological classification given above was

made completely in terms of the particle graph $g$. It is sometimes

useful to combine the particle graph $g(G)$ and the quark graph $G$ into

a single quark-particle graph $\bar{g}(G)$, in the way illustrated in Fig. 12
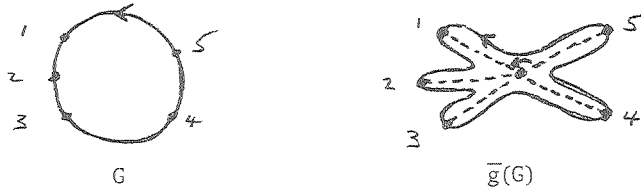


G                                    $\bar{g}(G)$

Figure 12. A graph $G$ and the quark-particle graph $\bar{g}(G)$ formed

from $G$. The particle lines of $g(G)$ are drawn as dotted lines in

$\bar{g}(G)$.

The orbits of $g(G)$ can be considered to be the independent

closed loops on the quark lines of $\bar{g}(G)$. Those closed quark-line

loops that pass through vertices are boundaries. Those that do

not are windows. The number of vertices and edges that occurs in

the Euler formula (3.20) for the genus is the number of vertices and

edges of the particle graph $g(G)$, which is a subgraph of $\bar{g}(G)$.

### 3.7 Topological Expansion

Each physical-region singularity of the S matrix is associated

with a Landau graph $g_L$. A formula for the discontinuity around

the singularity associated with graph $g_L$ is obtained by replacing

each vertex of $g_L$ by the corresponding scattering function[15,16].

This scattering function is specified by the set of edges incident

upon the vertex to which it corresponds. These edges can be

identified with the external edges of the particle graphs $g$

constructed above.

The topological expansion is the assumption that each scattering

function can be expressed as a sum of terms, one corresponding to

each of the different topological classes specified in the preceding

subsection. This expansion is required to be compatible with the

discontinuity formulas, in the sense that if the full expansion is

introduced into each of the scattering functions that occur in any

discontinuity equation, and the full equation is then decomposed

into terms of different topological class then the terms of each

class separately satisfy the equation: there is no cancellation

among the terms in the equation that have different topological

character. This assumption that the contributions to any discontinuity

equation corresponding to graphs of the same topological character should cancel among themselves has been discussed extensively before, in connection with the derivation of the discontinuity formulas[15,16,17].

## 3.8 The Zero-Entropy Functions

The validity of the topological expansion is assumed. Then the zero-entropy component of any discontinuity equation can be examined. Each scattering function is the sum of a pure ortho part plus a pure para part plus higher-order terms formed from products of ortho and para parts. These higher-order parts do not contribute to the zero-entropy component of the discontinuity equation. Thus the zero-entropy component separates into two parts, an ortho part and a para part, each of which must separately be satisfied, since each belongs, according to our classification scheme, to a separate topological class.

By virtue of the entropy property the zero-entropy terms can be formed only from zero-entropy factors. Thus the scattering function associated with each vertex of the Landau graph is replaced, in the ortho (para) zero-entropy component of the full discontinuity equation, simply by the zero-entropy ortho (para) amplitude. Consequently all the discontinuity equations for the zero-entropy ortho (para) amplitudes are identical to the discontinuity equations for the full scattering function with two exceptions: (1) the discontinuity is zero unless the Landau graph is planar; and (2) the full scattering amplitudes are replaced everywhere in the discontinuity equation by the corresponding ortho ( para ) amplitudes.

By virtue of the occurrence of only those singularities that correspond to planar Landau graphs the analytic structure of the ortho and para functions is much simpler than that of the full

scattering function. It is expected that these functions should have moving Regge poles but no Regge cuts. They should, in a first approximation, be similar to the Veneziano dual-resonance model functions,[18,19] with the addition of a spin-flavor structure, finite widths, and a planar singularity structure in momentum space.

## 3.9 Regge Recurrences

The property represented in (3.19) and Fig. 9 says the spinor factor in the zero entropy functions $A^O$ or $A^P$ has the pole-factorization property indicated in Fig. 13
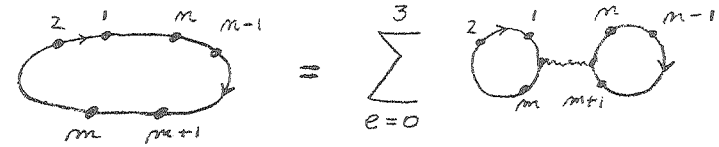


Figure 13. Pole factorization property.

Thus if $f^O(p)$ has a factorizable pole corresponding to a certain value $\ell \geq 1$ of angular momentum transferred between $(1, \ldots, m)$ and $(m + 1, \ldots, n)$ then the full function $A^O$ has factorizable poles corresponding to a set of four intermediate states, having total angular momentum values $J = \ell$ and $\ell + 1$, $\ell$, $\ell - 1$.

If $f^O(p)$ has Regge behavior of the kind exhibited by the Veneziano dual-resonance function[18], then for each factorizable pole of $f^O(p)$ corresponding to orbital angular momentum $\ell \geq 1$ there will be a quartet of factorizable poles of $A^O(p)$ corresponding to total angular momentum $\ell$ and $(\ell + 1, \ell, \ell - 1)$.

The function $f^O(p)$ is assumed to have a Regge pole with the lowest $\ell = 0$ pole identifiable with our external set of sixteen mesons $(\pi, \rho, \omega, \eta)$, which are assumed to be degenerate in the zero-entropy level. The higher values of $\ell$ will then generate recurrences

of the set $(\pi, \rho, \omega, \eta)$.

If the functions $f^0(p)$ and $f^p(p)$ are now generalized to represent the cases where the external particles are recurrences of the $\ell = 0$ mesons then one must include[19] for each external particle $i$ of angular momentum $\ell_i$ a set of $\ell_i$ vector indices, $\mu_1^{(i)} \ldots \mu_{\ell_i}^{(i)}$ that are such that $p_i \cdot f^0(p) = p_i \cdot f^p(p) = 0$ when the inner product is formed with any one of the indices $\mu_{\ell_j}^{(i)}$. Consequently, the earlier equation

$$f^p(p) = \mathscr{P}(f^0(p)) = f^0(\tilde{p}) = f^0(p) \qquad (3.27a)$$

becomes replaced by

$$f^p(p) = \mathscr{P}(f^0(p)) = (-1)^{\sum_{i=1}^{n} \ell_i} f^0(p) \qquad (3.27b)$$

When nonzero values of the $\ell_i$ are allowed there is also an extra factor of $(-1)^{\sum \ell_i}$ in the charge conjugation equation (3.10). This comes from a consideration of, for example, the two definitions of $\rho^+$ and $\rho^-$ implicit in Figs. 14a and 14b.
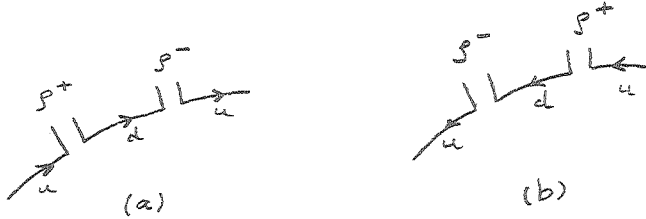


Figure 14. The normal quark structures of $\rho^+$ and $\rho^-$ is shown in (a), whereas (b) shows the definition induced by reversing the quark lines. If the quark wave function has angular momentum $\ell_i$ then the difference is represented by a factor $(-1)^{\ell_i}$.

In the discussion in §3.4 of isospin invariance there was no

change in the definitions of $\rho^+$ and $\rho^-$ of the kind shown in Fig. 14. However, the function $f^0(p)$ was changed due to a reversal of the order of the arguments. (See Fig. 4(c)) In the dual-resonance amplitude[19] this change induces a change $(-1)^{\sum \ell_i}$, and we assume that this property holds also for our function $f^0(p)$:

$$f^0(p_n, \ldots, p_1) = (-1)^{\sum \ell_i} f^0(p_1, \ldots, p_n) = \mathscr{P} f^0(p_1, \ldots, p_n). \qquad (3.28)$$

The fact that one gets the same factor $(-1)^{\sum \ell_i}$ by either reversing the direction of the quark arrow, as in Fig. 4, (a) → (b), (or Fig. 14, (a) → (b)) or by reversing the cyclic order of the vertices, as in Fig. 4, (a) → (c), means that the amplitude corresponding to a graph does not depend on how this graph is placed on the paper: the operation of turning over or reflecting a graph, as in Fig. 4, (b) → (c), does not alter the amplitude corresponding to it. Thus the equivalence of the two graphs of Fig. 2, or of Fig. 3, is maintained also for $\ell_i > 0$.

### 3.10 Suppression of Ortho-Para Mixing

Chew and Rosenzweig have discussed why the amplitudes corresponding to graphs $g$ with topological index $\gamma(g) > 0$ should be suppressed relative to the $\gamma(g) = 0$ contributions. A principal damping mechanism for the amplitudes corresponding to the nonplanar graphs comes from an averaging over neighboring poles along a Regge trajectory. In certain situations these contributions appear with a factor $(-1)^\ell$. If the remaining factors depend smoothly on $\ell$ then there should be strong suppression.

The present theory differs from that discussed by Chew and Rosenzweig by the inclusion of spin. This has led to the ortho

and para amplitudes, and then to contributions in which ortho amplitudes are coupled to para amplitudes.

If a line i of a graph g connects an ortho vertex to a para vertex then (3.27b) introduces a factor $(-1)^{\ell_i}$ into the corresponding amplitude. This should give a suppression analogous to the one discussed by Chew and Rosenzweig.

## 3.11 Parity Doublets

The theory has, by construction, no parity-doublet partners of the basic 16-plet $(\pi, \rho, \omega, \eta)$. Consider, then, the exact pole-factorization property

$$\text{Residue} \ (A^O + A^p + A^{h.o.}) = (A_1^O + A_1^p + A_1^{h.o.}) \cdot (A_2^O + A_2^p + H_2^{h.o.})$$

where $A^{h.o.}$ represents the higher order terms. $\quad$ (3.29)

Each amplitude $(A_i^O + A_i^p + A_i^{h.o.})$ conserves parity, with the parities defined in the way specified under Eq. (3.8). This is true for $A_i^O + A_i^p$, by explicit calculation, and will be true in general because of the general property that each contribution is constructed from _sums_ over ortho and para parts, not differences. Thus the initial and final states in (3.29) will have parities that are equal to the parities of the intermediate particles $(\pi, \rho, \omega, \eta)$, as defined under (3.8).

Consider, however, the zero-entropy part of (3.29):

$$\begin{aligned}
\text{Residue} \ (A^O + A^p) &= A_1^O A_2^O + A_1^p A_2^p \\
&= \left(\frac{A_1^O + A_1^p}{\sqrt{2}}\right)\left(\frac{A_2^O + A_2^p}{\sqrt{2}}\right) \\
&+ \left(\frac{A_1^O - A_1^p}{\sqrt{2}}\right)\left(\frac{A_2^O - A_2^p}{\sqrt{2}}\right)
\end{aligned}$$

(3.30)

The differences $(A_i^O - A_i^p)$ have, by virtue of the results obtained in §3.2, transitions only between states of opposite parity, provided the intrinsic parities are defined as under (3.8). Thus the initial and final states in the last term of (3.30) have opposite parity to the initial and final states of the other term. The normal way to describe this situation is to say that $(A_i^O - A_i^p)$ conserves parity, but couples to the parity-doublet partners of the $(\pi, \rho, \omega, \eta)$. In this sense the zero-entropy level of the topological expansion has parity doublets, even though these are not present in the full solution.

This situation emphasizes the fact that the zero-entropy approximation is expected to be a good approximation in situations where one can average over many values of $\ell$. The difference between the "correct" and the zero-entropy forms of the residue is

$$\Delta = \frac{1}{2}\left[(A_1^O + A_1^p) \cdot (A_2^O + A_2^p) - (A_1^O - A_1^p)(A_2^O - A_2^p)\right] = A_1^O A_2^p + A_1^p A_2^O \ .$$

This difference is expected to become small when averaged over many values of $\ell$, with a smoothly varying weighting factor. But for individual $\ell$ values it is large.

## 4. BARYONS

### 4.1 The Zero-Entropy Amplitudes

Earlier efforts[6-10] to include baryons in the topological expansion correspond to a picture of the baryon as a set of three quarks lying on the outer edges of three strips whose inner edges lie in close proximity to a single "dotted" line, called by various authors a dotted, junction, or mating line. This approach leads to apparently insurmountable difficulties with the analytic structure. A brief discussion of these difficulties is given in the appendix.

Analysis of these difficulties indicates that the zero-entropy structures for general hadronic processes must be essentially the same as those for meson processes, namely circular graphs. This can be achieved by considering some of the edges to represent diquarks. The theory then has mesons, baryons, antibaryons, and baryonium. Each meson vertex lies at the junction of two quark edges. Each baryonium vertex lies at the junction of two diquark edges. Each baryon and antibaryon vertex lies at the junction of a quark edge and a diquark edge.

Our convention is to run the quark and diquark edges in opposite directions around the circular graph, with a quark and diquark edge running into each baryon vertex and out of each anti baryon vertex. The quark and diquark edges are drawn as thin and thick lines, respectively.

The rules for forming the ortho and para amplitudes are essentially the same as in the meson sector, with the diquark propagator being the symmetrized form of two quark propagators. For example, in the ortho amplitudes the diquark propagator is

$$D^o {}^{\lambda_1 \lambda_2}_{\alpha_1 \alpha_2} {}^{\delta_1 \delta_2}_{\mathring{\beta}_1 \mathring{\beta}_2}$$

$$= \frac{1}{2} \left[ u_a \cdot \sigma_{\alpha_1 \mathring{\beta}_1} \, u_a \cdot \sigma_{\alpha_2 \mathring{\beta}_2} \, \delta^{\lambda_1 \delta_1} \delta^{\lambda_2 \delta_2} \right.$$

$$\left. + u_a \cdot \sigma_{\alpha_1 \mathring{\beta}_2} \, u_a \cdot \sigma_{\alpha_2 \mathring{\beta}_1} \, \delta^{\lambda_1 \delta_2} \delta^{\lambda_2 \delta_1} \right], \quad (4.1)$$

where $u_a = p_a/m_a$ is formed from the mathematical momentum-energy vector $p_a$ of the particle associated with the vertex that lies at the leading end of the diquark edge. The para propagator $D^p$

is the same with $u_a$ replaced by $-u_b = -p_b/m_b$, where $p_b$ is the mathematical momentum-energy vector of the particle associated with the vertex that lies on the trailing end of the diquark edge. The indices $\lambda_i$ and $\delta_i$ are flavor indices.

For each baryon in a scattering function there is a set of arguments $(^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3})$ consisting of an ordered triplet of pairs $(\alpha_i, \lambda_i)$ of spin-flavor indices. The labelling in (4.1) corresponds to assigning the first two pairs to the diquark and the final pair $(\alpha_3, \lambda_3)$ to the quark. This particular way of labelling corresponds to one contribution to $A^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3}$. This contribution is called $A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3}$ whereas the other two are called $A^{(1)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3}$ and $A^{(2)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3}$. The superscript (e) designates the position (1, 2, or 3) of the pair of spin-flavor indices that are assigned to the quark. The three contributions are represented graphically in Fig. 16.
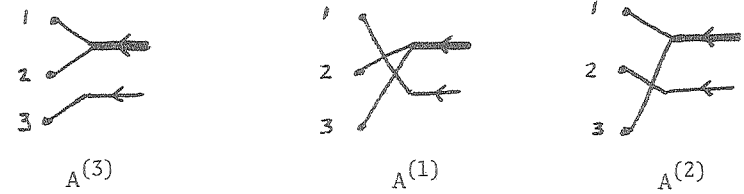


Figure 16. Three contributions to $A^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3}$. In contribution $A^{(e)}$ the quark is labelled by the pair of arguments $(\alpha_i, \lambda_i)$ that stand in the position e.

This sum over the three ways of labelling the quark is made independently for each baryon and antibaryon. Thus letting b be an index that labels the baryons and antibaryons that contribute to a process, and letting $P^b_+$ and $P^b_-$ be the operators that effect the cyclic permutations $(1 \to 2 \to 3 \to 1)$ and $(1 \to 3 \to 2 \to 1)$,

respectively, on the three quark variables associated with baryon or antibaryon  b, one may write the full amplitude  A  as

$$A = \prod_b \left( \frac{I^b + P_+^b + P_-^b}{\sqrt{3}} \right) (A^{o(3)} + A^{p(3)}), \qquad (4.2)$$

where  $I^b$  is the identity operator in the space of the three quark variables associated with baryon or antibaryon  b,  and  $A^{o(3)}$  and  $A^{p(3)}$  are the ortho and para amplitudes in which the quarks are labelled, for every  b, by the arguments standing in the third position  in the triad  $\begin{pmatrix} \lambda_1^b & \lambda_2^b & \lambda_3^b \\ \alpha_1^b & \alpha_2^b & \alpha_3^b \end{pmatrix}$.

The amplitude  A  associated with the  $\ell$th  orbital recurrence of baryon  b  has a term that is represented by attaching a set  $(\mu_1, \ldots, \mu_\ell)$  of vector arguments to  A.  In the case of a baryon or antibaryon . each of these vector arguments is associated with a two-dimensional representation space of the group  $S_3$  of permutations of three objects.  Letting the two-valued index that labels the components in this space be denoted by  $\sigma_i \epsilon$  (1, 2)  the action of  $P_+^b$  and  $P_-^b$  on  $A^{(3)} \equiv A^{o(3)} + A^{p(3)}$  is  represented as  follows: (the super scripts  b  are now suppressed)

$$P_+ \ A^{(3)} \begin{matrix} \lambda_1 \lambda_2 \lambda_3 \\ \alpha_1 \alpha_2 \alpha_3 \end{matrix} (\mu_1, \sigma_1; \mu_2, \sigma_2; \ \cdots \ ; \mu_\ell, \sigma_\ell : p)$$

$$= \sum_{\substack{\sigma_i'=1 \\ i=1 \ldots, \ell}}^{2} \ A^{(3)} \begin{matrix} \lambda_3 \lambda_1 \lambda_2 \\ \alpha_3 \alpha_1 \alpha_2 \end{matrix} (\mu_1, \sigma_1'; \mu_2, \sigma_1'; \ \cdots \ ; \mu_\ell, \sigma_\ell' : p) \ \prod_{i=1}^{\ell} (P_+)_{\sigma_i \sigma_i'}$$

$$(4.3a)$$

$$P_- \ A^{(3)} \begin{pmatrix} \lambda_1 \lambda_2 \lambda_3 \\ \alpha_1 \alpha_2 \alpha_3 \end{pmatrix} (\mu_1, \sigma_1; \mu_2, \sigma_2; \ \cdots \ ; \mu_\ell, \sigma_\ell : p)$$

$$= \sum_{\substack{\sigma_i'=1 \\ i=1, \ldots, \ell}}^{2} \ A^{(3)} \begin{matrix} \lambda_2 \lambda_3 \lambda_1 \\ \alpha_2 \alpha_3 \alpha_1 \end{matrix} (\mu_1, \sigma_1'; \mu_2, \sigma_2'; \ \cdots \ ; \mu_\ell, \sigma_\ell' : p)$$

$$\prod_{i=1}^{\ell} (P_-)_{\sigma_i \sigma_i'} \qquad (4.3b)$$

where  $(P_+)_{\sigma_i \sigma_i'}$  and  $(P_-)_{\sigma_i \sigma_i'}$  are the two-dimensional matrices that represent the  permutations  $P_+$  and  $P_-$,  respectively.

The two-dimensional representation-space associated with each index  $\mu_i$  can be considered to arise from the fact that the indices  $\mu_i$  refer to the angular momentum of a three-particle system.  In a constituent-quark  model these vectors would be constructed from the vectors[20]

$$\rho^\mu = \frac{1}{\sqrt{2}} \ (r_1 - r_2)^\mu \qquad (4.4a)$$

and

$$\lambda^\mu = \frac{1}{\sqrt{2}} \ (2r_3 - r_1 - r_2)^\mu \ , \qquad (4.4b)$$

which are basis vectors  of the two-dimensional representation of the group of permutation on three objects.

To understand the significance of this two-dimensional representation space consider a simple constituent quark model for the baryon. Then our amplitude  $A^{(3)}$  would be constructed as follows:

$$A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (\mu_1, \sigma_1; \ldots ; \mu_\ell, \sigma_\ell; p)$$

$$= \int d^3 r_1 d^3 r_2 d^3 r_3 \ \delta^3 (r_1 + r_2 + r_3)$$

$$A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p) \ (r_1^2 + r_2^2 + r_3^2)^{-\ell/2}$$

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} (r_1 - r_2) \\ \frac{1}{\sqrt{6}} (2r_3 - r_1 - r_2) \end{pmatrix}_{\mu_1} \ \cdots \ \begin{pmatrix} \frac{1}{\sqrt{2}} (r_1 - r_2) \\ \frac{1}{\sqrt{6}} (2r_3 - r_1 - r_3) \end{pmatrix}_{\mu_\ell} \right\} \quad (4.5)$$

where the upper (or lower) part of the $i$th vector factor is used if $\sigma_i$ is 1 (or 2). The upper vector $\frac{1}{\sqrt{2}}(r_1 - r_2)$ gives the angular momentum of the two quarks of the diquark, whereas the lower vector gives the angular momentum of the quark-diquark system[20].

To fix ideas suppose, for example, that $A^{(3)} (r_1, r_2, r_3 : p)$ depends on $r_1$ and $r_2$ only through the combination $(r_1 - r_2)^2$. And suppose $\ell = 1$. Then the integration in (4.5) will give zero for the upper component labelled by $\sigma_1 = 1$, and the function $A^{(3)}(\mu_1, \sigma_1)$ will represent the situation in which the orbital angular momentum of the quarks in the diquark is zero, whereas that of the quark-diquark system is one.

The variables $(\alpha_3, \lambda_3, r_3)$ occupying the third position in the arguments of $A^{(3)}$ play a distinguished role: they label the quark. The function $A^{(1)}$ is supposed to be the same function, but with variables lying in the first position playing this distinguished role:

$$A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p)$$

$$= \ A^{(1)}{}^{\lambda_3 \lambda_1 \lambda_2}_{\alpha_3 \alpha_1 \alpha_2} \quad (r_3, r_1, r_2 : p) \ . \quad (4.6)$$

And the function $\sqrt{3}A$ is the sum:

$$\sqrt{3} \ A^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p) \ =$$

$$A^{(1)\lambda_1 \lambda_2 \lambda_3}_{\ \ \alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p)$$

$$+ \ A^{(2)\lambda_1 \lambda_2 \lambda_3}_{\ \ \alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p)$$

$$+ \ A^{(3)\lambda_1 \lambda_2 \lambda_3}_{\ \ \alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p) \quad (4.7)$$

$$= \ A^{(3)}{}^{\lambda_2 \lambda_3 \lambda_1}_{\alpha_2 \alpha_3 \alpha_1} (r_2, r_3, r_1 : p)$$

$$+ \ A^{(3)}{}^{\lambda_3 \lambda_1 \lambda_2}_{\alpha_3 \alpha_1 \alpha_2} (r_3, r_1, r_2 : p)$$

$$+ \ A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} (r_1, r_2, r_3 : p)$$

$$= \ (P_- + P_+ + I) \ A^{(3)\lambda_1 \lambda_2 \lambda_3}_{\ \ \alpha_1 \alpha_2 \alpha_3} (r_1, r_2, r_3 : p) \quad (4.8)$$

These permutations $P_+$ and $P_-$ act on both the spin-flavor arguments $(\alpha_i, \lambda_i)$ and also on the arguments $r_i$. These latter transformations are represented by transformations in the two-dimensional space spanned by $\frac{1}{\sqrt{2}} (r_1 - r_2)$ and $\frac{1}{\sqrt{6}} (2r_3 - r_1 - r_2)$.

$$A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (\mu_1, \sigma_1; \ldots ; \mu_\ell, \sigma_\ell : p)$$

$$= \int d^3 r_1 d^3 r_2 d^3 r_3 \quad \delta^3 (r_1 + r_2 + r_3)$$

$$A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p) \quad (r_1^2 + r_2^2 + r_3^2)^{-\ell/2}$$

$$\left\{ \left( \begin{array}{c} \frac{1}{\sqrt{2}} (r_1 - r_2) \\ \frac{1}{\sqrt{6}} (2r_3 - r_1 - r_2) \end{array} \right)_{\mu_1} \quad \cdots \quad \left( \begin{array}{c} \frac{1}{\sqrt{2}} (r_1 - r_2) \\ \frac{1}{\sqrt{6}} (2r_3 - r_1 - r_3) \end{array} \right)_{\mu_\ell} \right\} \quad (4.5)$$

where the upper (or lower) part of the $\underline{i}$th vector factor is used if $\sigma_i$ is 1 (or 2). The upper vector $\frac{1}{\sqrt{2}}(r_1 - r_2)$ gives the angular momentum of the two quarks of the diquark, whereas the lower vector gives the angular momentum of the quark-diquark system[20].

To fix ideas suppose, for example, that $A^{(3)}$ $(r_1, r_2, r_3 : p)$ depends on $r_1$ and $r_2$ only through the combination $(r_1 - r_2)^2$. And suppose $\ell = 1$. Then the integration in (4.5) will give zero for the upper component labelled by $\sigma_1 = 1$, and the function $A^{(3)}(\mu_1, \sigma_1)$ will represent the situation in which the orbital angular momentum of the quarks in the diquark is zero, whereas that of the quark-diquark system is one.

The variables $(\alpha_3, \lambda_3, r_3)$ occupying the third position in the arguments of $A^{(3)}$ play a distinguished role: they label the quark. The function $A^{(1)}$ is supposed to be the same function, but with variables lying in the first position playing this distinguished role:

$$A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p)$$

$$= A^{(1)}{}^{\lambda_3 \lambda_1 \lambda_2}_{\alpha_3 \alpha_1 \alpha_2} \quad (r_3, r_1, r_2 : p) . \quad (4.6)$$

And the function $\sqrt{3} A$ is the sum:

$$\sqrt{3} \, A^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p) \quad =$$

$$A^{(1)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p)$$

$$+ \, A^{(2)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p)$$

$$+ \, A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p) \quad (4.7)$$

$$= \, A^{(3)}{}^{\lambda_2 \lambda_3 \lambda_1}_{\alpha_2 \alpha_3 \alpha_1} (r_2, r_3, r_1 : p)$$

$$+ \, A^{(3)}{}^{\lambda_3 \lambda_1 \lambda_2}_{\alpha_3 \alpha_1 \alpha_2} \quad (r_3, r_1, r_2 : p)$$

$$+ \, A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} \quad (r_1, r_2, r_3 : p )$$

$$= \, (P_- + P_+ + I) \, A^{(3)}{}^{\lambda_1 \lambda_2 \lambda_3}_{\alpha_1 \alpha_2 \alpha_3} (r_1, r_2, r_3 : p) \quad (4.8)$$

These permutations $P_+$ and $P_-$ act on both the spin-flavor arguments $(\alpha_i, \lambda_i)$ and also on the arguments $r_i$. These latter transformations are represented by transformations in the two-dimensional space spanned by $\frac{1}{\sqrt{2}} (r_1 - r_2)$ and $\frac{1}{\sqrt{6}} (2r_3 - r_1 - r_2)$.

To see this clearly suppose that the function

$A^{(3)}{}^{\lambda_1\lambda_2\lambda_3}_{\;\;\alpha_1\alpha_2\alpha_3}(r_1, r_2, r_3: p)$  has the form

$$F^{(3)}{}^{\lambda_1\lambda_2\lambda_3}_{\;\;\alpha_1\alpha_2\alpha_3} \quad \delta^3(r_3 - r(p)) \; \delta((r_1 - r_2)^2 - c^2) \tag{4.9}$$

Then

$$A^{(3)}{}^{\lambda_1\lambda_2\lambda_3}_{\;\;\alpha_1\alpha_2\alpha_3} \quad (\mu_1, \sigma_1: p)$$

$$= \begin{cases} 0 & \text{if } \sigma_1 = 1 \\[2em] \text{const. } (r(p))_{\mu_1} & \text{if } \sigma_1 = 2 \end{cases} \tag{4.10}$$

The function  $A^{(1)}$  would be  identical if expressed in the basis

$$\left\{ \begin{array}{c} \dfrac{1}{\sqrt{2}} (r_2 - r_3)_\mu \\[1.5em] \dfrac{1}{\sqrt{6}} (2r_1 - r_2 - r_3)_\mu \end{array} \right\} \tag{4.11}$$

The result in this basis can be transformed into the original basis by a transformation representing a cycle permutation.

The most convenient basis for this purpose is the one in which $P_+$  and  $P_-$  are diagonal.  The two basis vectors in this representation are

$$\frac{1}{\sqrt{3}} (r_3 + q \, r_1 + \bar{q} \, r_2)$$

and

$$\frac{1}{\sqrt{3}} (r_3 + \bar{q} \, r_1 + q \, r_2) \, , \tag{4.12}$$

where

$$q = (-1)^{1/3} = -\frac{1}{2} + i \frac{\sqrt{3}}{2} \tag{4.13a}$$

and

$$\bar{q} = q^2 = -\frac{1}{2} - i \frac{\sqrt{3}}{2} \, . \tag{4.13b}$$

In this representation

$$P_+ = \begin{pmatrix} q & 0 \\ 0 & \bar{q} \end{pmatrix} \tag{4.14a}$$

and

$$P_- = \begin{pmatrix} \bar{q} & 0 \\ 0 & q \end{pmatrix} \, . \tag{4.14b}$$

In this representation the vector  $\dfrac{1}{\sqrt{6}} (2r_3 - r_1 - r_2)$  is represented by

$$\psi^{(3)} \equiv \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \sim \frac{1}{\sqrt{6}} (2r_3 - r_1 - r_2) \tag{4.15}$$

Thus

$$P_+ \psi^{(3)} = \begin{pmatrix} q/\sqrt{2} \\ \bar{q}/\sqrt{2} \end{pmatrix} \sim \frac{1}{\sqrt{6}} (2r_2 - r_1 - r_3)$$

$$\equiv \psi^{(2)} \tag{4.16}$$

and

$$P_- \psi^{(3)} = \begin{pmatrix} \bar{q}/\sqrt{2} \\ q/\sqrt{2} \end{pmatrix} \sim \frac{1}{\sqrt{6}} (2r_1 - r_2 - r_3)$$

$$\equiv \psi^{(1)} \tag{4.17}$$

imbedded. Then for graphs $g$ of topological index $\gamma(g) = 0$ both the particle graph $g$ and the quark-diquark graph $G$ are planar, as before.

The quark-diquark lines of $\bar{g}(G)$ never cross each other in the imbedding surface. Neither do the particle lines. However, a quark line may cross a particle line, in the way shown in Fig. 18. This "cross-over" constitutes an element of topological complexity, and is excluded, by definition, from the zero-entropy graphs.

The cross-over quark line in the cross-over diagram of Fig. 18 separates the right-hand part of the strip from the left-hand part. This transition is treated in exactly the same way as an ortho-para transition: in the topological classification the graph $\bar{g}(G)$ is cut along the cross-over quark line. Thus each part $\bar{g}(G)_i$ of the full graph has no ortho-para transition and no cross-over transition.

The surface associated with a general hadron graph $\bar{g}(G)$ is not always orientable. A simple non orientable graph $\bar{g}(G)$ is shown in Fig. 19.
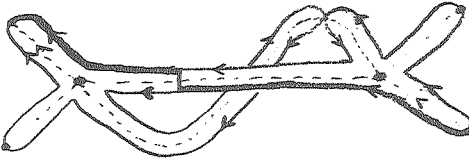


Figure 19. A graph $\bar{g}(G)$ with non orientable surface.

The genus of this graph is

$$h(g) = \frac{e(g) - v(g) - f(g) + 2}{2}$$

$$= \frac{6 - 6 - 1 + 2}{2} = \frac{1}{2}$$

The half-integer value corresponds to the presence of a "cross-cap", which can occur on a non oriented surface.

The topological index $\gamma(\bar{g})$ of the graph $\bar{g}(G)$ is

$$\gamma(\bar{g}) = \beta(\bar{g}) - \omega(\bar{g})$$

$$= \beta(g) - 1$$

The number of closed loops $\beta(\bar{g})$ is the number of independent closed loops in the particle graph represented by the dotted lines of $\bar{g}$. The number of windows $\omega(\bar{g})$ is the number of orbits of $\bar{g}(G)$ that do not pass through any vertex. The entropy property and its proof are the same as in the meson case.

4.3 Orbital Suppression Factors

The ortho-para transitions are suppressed by the factor $(-1)^\ell$ as in the meson case.

The orbital angular momentum $\ell$ is now assumed to reside in the quark-diquark system. Then the dependence of the amplitudes $A^{(3)}$ on each of the indices $\sigma_i$ is given by the vector $\psi^{(3)}$ defined in (4.15). Thus (4.18) and (4.19) give an explicit orbital angular-momentum factor

$$(\psi^{(3)} | (I + P_+ + P_-) | \psi^{(3)}). \tag{4.20}$$

Equations (4.13) and (4.14) give

$$(\psi^{(3)} | I | \psi^{(3)}) = 1 \tag{4.21a}$$

$$(\psi^{(3)} | P_+ | \psi^{(3)}) = -\frac{1}{2} \tag{4.21b}$$

$$(\psi^{(3)} \mid P_- \mid \psi^{(3)}) = -\frac{1}{2} . \qquad (4.21c)$$

There is one of these factors for each one of the $\ell$ indices $\mu_i$ in (4.18). The combination of (4.21) with the ortho-para factor of $(-1)^\ell$ gives the explicit orbital factors indicated in Fig. 20
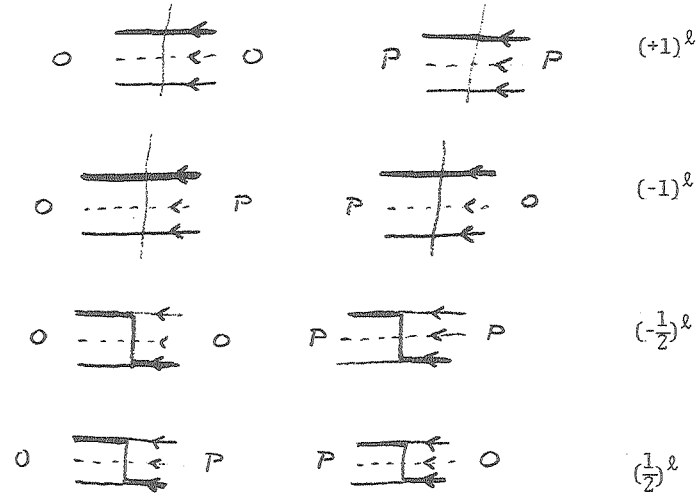


Figure 20. Orbital Suppression Factors.

Thus for large $\ell$ there is, in an average sense, an explicit suppression of both the ortho-para and cross-over transitions. Hence the dominant large $\ell$ behavior should be the one in which there is no ortho-para mixing and no diquark fragmentation (i.e. no cross-overs) Thus at large $\ell$ the physical Regge trajectories should approach the trajectories of the zero-entropy amplitudes.

## 4.4 Particle Spectra

To represent within the general S-matrix context the requirement of symmetry under the permutations of the variables pertaining to the

three quarks in each baryon a two-dimensional representation space of the permutation group was introduced in association with each of the angular momentum indices $\mu_i (i = 1, \ldots , \ell)$ associated with each baryon or antibaryon recurrence. This representation space can be understood on the basis of a constituent quark model. However, it reflects only certain group-theoretic aspects of such a model, not its dynamical aspects. The dynamics in the present theory is represented, as usual in S-matrix theory, by the combination of the on-mass-shell requirements of unitarity, analyticity, duality, and invariance, where the invariances now includes not only Lorentz invariance but also permutation invariance on the three indices associated with each baryon, as defined by (4.3).

Since the permutation group theoretic structure is the same as that of the constituent-quark model, with all orbital angular momentum in the quark-diquark separation, the theory yields the usual particle spectra at the lowest $\ell$-values, and in particular the usual $(56, 0^+)$ and $(70, 1^-)$ multiplets. The zero-entropy equations should generate a single Regge trajectory corresponding to these two multiplets. If the various suppression mechanisms are indeed operative then the higher-order corrections to these trajectories should be suppressed at high $\ell$-values, and hence the physical even and odd trajectories should merge at high $\ell$. An important test of the theory will be the calculation of the separations of these trajectories as one moves to lower $\ell$ values. The corresponding calculation in the vector meson sector were successful[2].

Another important test will be the calculation of the separation
between the vector and pseudoscalar meson trajectories. A third
important test will be the calculation of the $\rho^0 \to \pi^+\pi^-$ decay rate.

Appendix.Failure of the Earlier Schemes

In the earlier baryon schemes[6-10] the product represented by
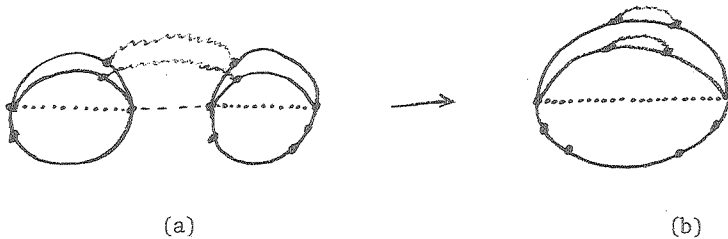the graphs of Fig. 21 was classified as zero entropy.



(a)                              (b)

Figure 21. A product classified as zero entropy by the earlier baryon
topological expansions schemes. The quarks, mesons, and baryons are
represented by solid, wiggly, and dashed lines, respectively. The
dotted lines are called "dotted" "junction", or "mating" lines by
different authors.

The zero-entropy amplitude represented by the quark graph on the RH
side of Fig. 21a has a pole singularity, and the discontinuity
represented by Fig. 21a has,consequently, an ice-cream-cone
diagram singularity represented by the graphs of Fig. 22.

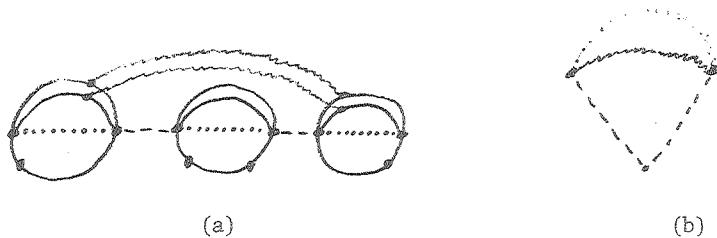

(a)                              (b)
Fig. 22. Graphs representing an ice-cream cone diagram singularity.

This singularity is also classified as zero entropy, and hence
this singularity should be present in the zero-entropy amplitude.

This singularity can be followed analytically into the crossed
channel that corresponds to reading Fig. 22(b) from bottom to top.
However,Fig. 22(a) shows that the singularity corresponding to the
top part of Fig. 22(b) has nonzero entropy, and hence does not lie
in the zero-entropy part of the corresponding amplitude. Thus a zero-
entropy ice-cream-cone diagram singularity lies in a part of the crossed-
channel normal-threshold discontinuity that is formed by combi-
ning a zero-entropy amplitude with a non zero-entropy amplitude. This
constitutes a failure of the homogeneity property of zero-entropy
amplitudes whereby all zero-entropy singularities are obtained by
using only the zero-entropy parts of the amplitudes that occur in the
various physical discontinuity formulas.

Any attempt to disrupt the usual tightly knit crossing structure
by putting crossed-channel singularities obtained by analytic
continuation of a zero-entropy amplitude into some other term of
the topological expansion would surely preclude the possibility of
any natural solution to the zero-entropy equations. On the other
hand, the inclusion of an amplitude with nonzero entropy into the
crossed-channel normal-threshold discontinuity equation for a zero-
entropy amplitude would destroy the closure property of the
equations for the zero-entropy functions.

## ACKNOWLEDGEMENTS

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